4. On Poisson Integrals

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1. Let $f(t)$ be an integrable function on the interval $[-\pi, \pi]$, then we can consider the Poisson integral

$$(1)\qquad u(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\frac{1-r^2}{1+r^2-2r\cos(t-\theta)}dt \quad (0\le r<1,\ 0\le\theta<2\pi).$$

The following theorem concerning the Poisson integral is well known: *if $f(t)$ has a derivative at $t=\theta_0$, then we have* $\lim_{r\to1}\frac{\partial u(re^{i\theta_0})}{\partial\theta}=f'(\theta_0)$. The purpose of this paper is to investigate whether this theorem holds for other derivatives. As

$$(2)\qquad \frac{\partial u(re^{i\theta})}{\partial\theta}=\frac{-1}{2\pi}\int_{-\pi}^{\pi}f(t)\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\right)dt,$$

we shall consider the integrals of this type.

2. We shall begin with the positive result.

THEOREM 1. *If $f(t)$ has a symmetric Borel derivative*[1] *at θ_0, then we have* $\lim_{r\to1}\frac{\partial}{\partial\theta}u(re^{i\theta_0})=B_s'f(\theta_0)$.

Proof. Without loss of generality, we can assume that $\theta_0=0$ and $B_s'f(\theta_0)=0$. If we set $F(t)=\int_t^{\pi}\frac{f(t)-f(-t)}{2t}dt$, $F(h)=F(0)+h\varepsilon(h)$, it follows from the hypothesis that for every $\varepsilon>0$ there exists δ such that $0\le h<\delta$ implies $|\varepsilon(h)|<\varepsilon$. Fixing δ we divide the integral (2) into three parts:

$$\frac{-1}{2\pi}\left[\int_{-\pi}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{\pi}\right]=\frac{-1}{2\pi}(I_1+I_2+I_3).$$

Integration by parts leads to the evaluation of I_3,

$$|I_3|\le M\cdot\frac{1-r}{4r\sin^4\delta/2}+M\int_{\delta}^{\pi}\left|\frac{\partial^2}{\partial t^2}\left(\frac{1-r^2}{1+r^2-2r\cos t}\right)\right|dt\le K(1-r),$$

where $M=\int_{-\pi}^{\pi}|f(t)|dt$, K is a constant not depending on r. Therefore

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1) A function $f(t)$ has a Borel derivative $\alpha(\neq\infty)$ at θ_0 if $\lim_{h\to0}\frac{1}{h}\int_0^h\frac{f(t+\theta_0)-f(\theta_0)}{t}dt=\alpha$ and we write it $B'f(\theta_0)$. Similarly $f(t)$ has a symmetric Borel derivative $B_s'f(\theta_0)=\alpha$ at θ_0 if $\lim_{h\to0}\frac{1}{h}\int_0^h\frac{f(\theta_0+t)-f(\theta_0-t)}{2t}dt=\alpha$, where the integrals are taken in the sense of $\lim_{\varepsilon\to0}\int_\varepsilon^h$.

$\lim_{r\to 1} I_3=0$, similarly $\lim_{r\to 1} I_1=0$. As for I_2, setting $P_r(t)=\dfrac{1-r^2}{1+r^2-2r\cos t}$,

$$I_2=\int_0^\delta \frac{f(t)-f(-t)}{2t}2t\frac{\partial}{\partial t}P_r(t)dt$$
$$=F(\delta)\frac{4r(1-r^2)\delta\sin\delta}{(1+r^2-2r\cos\delta)^2}+\int_0^\delta F(t)\frac{\partial}{\partial t}\Big(2t\frac{\partial}{\partial t}P_r(t)\Big)dt$$
$$=o(1)^{2)}+\int_0^\delta F(0)\frac{\partial}{\partial t}\Big(2t\frac{\partial}{\partial t}P_r(t)\Big)dt+\int_0^\delta t\varepsilon(t)\frac{\partial}{\partial t}\Big(2t\frac{\partial}{\partial t}P_r(t)\Big)dt.$$

The second term is $o(1)$, and the last term I_2' is divided into two terms: $$I_2'=\int_0^\delta 2\varepsilon(t)\,t\frac{\partial}{\partial t}P_r(t)dt+\int_0^\delta 2\varepsilon(t)\,t^2\frac{\partial^2}{\partial t^2}P_r(t)dt.$$

Since $\displaystyle\int_0^\delta\Big|t\frac{\partial}{\partial t}P_r(t)\Big|dt$, $\displaystyle\int_0^\delta\Big|t^2\frac{\partial^2}{\partial t^2}P_r(t)\Big|dt$ are bounded in r, we can see $|I_2'|\le\varepsilon K_1$, where K_1 is a constant not depending on r. Collecting the results we have $\displaystyle\lim_{r\to 1}\frac{\partial u(r)}{\partial\theta}=0=B_s'f(0)$. Q.E.D.

Instead of the Borel derivative, if we take up the approximate derivative[3] this theorem does not hold in general. For example, let $f(t)$ be defined in $[-\pi,\pi]$ as follows:

$$f(t)=\begin{cases}1 & \text{for } t\in I_n=[1/2^n,\ 1/2^n+1/4^n],\ n=1,2,\cdots,\\ 0 & \text{for } t\in[-\pi,\pi]-\bigcup_{n=1} I_n,\end{cases}$$

$f(t)$ is approximately derivable at $t=0$ and $f'_{ap}(0)=0$,[3] but $\displaystyle\overline{\lim_{r\to 1}}\frac{\partial u(r)}{\partial\theta}>0$. In fact if we set $r_n=1-1/2^n$ $(n=1,2,\cdots)$, $\dfrac{\partial u(r_n)}{\partial\theta}$ always exceed $(5\pi^3)^{-1}$.

3. In the preceding section we have studied that the approximate derivative is too weak to restrict the boundary behaviour of $\dfrac{\partial u}{\partial\theta}$. Now we are faced with the problem, how can we expect the positive result in this direction? As a trial, we shall define a new derivative which is based on an approximate derivative but has an order.

Let x_0 be a real number, E be a set of real numbers and $\alpha\ge 1$. Setting $I_h=[x_0,x_0+h]$ $(I_h=[x_0+h,x_0])$ for $h>0$ $(h<0)$, if we have $\lim_{h\to 0}\text{mes.}(E\cdot I_h)/(\text{mes.}I_h)^\alpha=0$ then we shall call x_0 is *a right-hand (left-hand) point of dispersion of order α for a set E.* If x_0 is simultaneously a right-hand and a left-hand point of dispersion of order α for E, it is called merely *a point of dispersion of order α for a set E.* Given a finite measurable function $f(t)$, for $\varepsilon>0$ and for τ we shall set $E(\varepsilon,\tau;x_0)=E\Big[x;\Big|\dfrac{f(x)-f(x_0)}{x-x_0}-\tau\Big|\ge\varepsilon\Big]$. For every $\varepsilon>0$, if x_0 is

2) This notation means that this term tends to zero as $r\to 1$.

3) Cf. S. Saks: Theory of the Integral, pp. 218–220.

a point of dispersion of order α for $E(\varepsilon, \tau; x_0)$, we shall say τ is *the approximate derivative of order α of $f(x)$ at x_0* and denote it $\tau = f_{ap}^{[\alpha]}(x_0)$.

Obviously if $f(x)$ is derivable in the usual sense at x_0 then $f'(x_0) = f_{ap}^{[\alpha]}(x_0)$ for every $\alpha \geq 1$, and if $f(x)$ has an approximate derivative of order α at x_0 and $\alpha \geq \alpha'$ then $f(x)$ has an approximate derivative of order α' at x_0 and $f_{ap}^{[\alpha]}(x_0) = f_{ap}^{[\alpha']}(x_0)$, and finally the concept of an approximate derivative of order 1 coincides with that of the usual approximate derivative.

As for the relation between the above defined ordered approximate derivative and the Borel derivative we shall show the following example[4] which permits us, for every $\alpha \geq 1$, to construct a set of positive measure P and an integrable function $F(x)$ such that there exists an approximate derivative of order α at *every* point of P but there exists Borel derivative at *no* point of P.

We can assume that α is a positive integer. For $k = 1, 2, \cdots$, we shall define the integer n_k in the following manner:

"n_k is the minimum number n such that

$$1 + 1/2 + 1/3 + \cdots + 1/n > 2^{(2\alpha-1)k+\alpha}\text{"}.$$

Next, we shall make two groups of intervals in [0, 1] according to the following steps.

[1] we shall divide the interval [0, 1] into $2n_1$ equal segments and denote the points of subdivision from left to right, $c_1, c_2, \cdots, c_{2n_1-1}$. Denoting δ_i the open interval of which center is c_i and has length $1/8n_1$ and δ'_i the open interval of the same center as δ_i and of length $(1/8n_1)^\alpha$, we shall call the former the intervals of 1st *group* 1st *class* and the latter the intervals of 1st *group* 2nd *class*.

[2] Removing from [0, 1] all intervals of 1st group 1st class we divide each remaining intervals into $2n_2$ equal segments whose terminal points are $c_{2n_1}, c_{2n_1+1}, \cdots$. As in [1] we describe two classes of intervals each of which has c_i as a center and is of length respectively $1/32n_1n_2$ and $(1/32n_1n_2)^\alpha$, and call them respectively the intervals of 2nd *group* 1st *class* and of 2nd *group* 2nd *class*.

[3] In general, the intervals of kth *group* are defined in the following: removing the all intervals of 1st class up to $(k-1)$th group we divide each remaining intervals into $2n_k$ equal segments. The points of this subdivision are the centers of intervals of 1st and of 2nd class, the former have length $1/(2^{2k+1}n_1n_2\cdots n_k)$, the latter $1/(2^{2k+1}n_1n_2\cdots n_k)^\alpha$.

Proceeding as is shown in the above steps, we shall obtain the intervals of kth group 1st class and 2nd class for every k. Removing from [0, 1] all the intervals of 1st class of each group, we obtain a perfect

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4) This is based on the example of Khintchine: A. Khintchine: Recherches sur la structure de fonctions mesurables, Fund. Math., **9**, 233 (1927).

set P_1. The set of all points of density for P_1 is denoted by P. As is easily seen mes. $P>1/2$, and this is the desired set. The desired function is now defined as

$$F(x)=\begin{cases}(n_1n_2\cdots n_k)^{\alpha-1} & \text{for } x \text{ which belongs to the intervals of } k^{\text{th}} \text{ group } 2^{\text{nd}} \text{ class, } k=1,2,\cdots,\\ 0 & \text{elsewhere.}\end{cases}$$

At each point x of P we have $F_{ap}^{[\alpha]}(x)=0$, whereas $F(x)$ has no derivative in the sense of Borel, and finally $F(x)$ is integrable in $[0, 1]$.

4. Letting $f(t)$ be a *bounded* measurable function on $[-\pi, \pi]$ and $\sup\limits_{-\pi\le t\le\pi}|f(t)|=M$, we shall consider the Poisson integral (1) in the first section. Concerning this we can state the following theorem.

THEOREM 2. *If $f(t)$ has an approximate derivative of order α at θ_0 for $\alpha>4$, we can obtain* $\lim\limits_{r\to1}\dfrac{\partial u(re^{i\theta_0})}{\partial\theta}=f_{ap}^{[\alpha]}(\theta_0)$.

Proof. As in Theorem 1, we may assume $\theta_0=0$, $f_{ap}^{[\alpha]}(\theta_0)=0$ and the integral (2) which expresses $\dfrac{\partial u(r)}{\partial\theta}$ is divided into three parts, however in this case δ is not a constant but depends on r, that is, we choose δ such that $\delta=\delta(r)=(1-r)^{2/\alpha}$.

Since $I_3=\int_\delta^\pi f(t)\dfrac{\partial}{\partial t}P_r(t)dt$ and $\dfrac{\partial}{\partial t}P_r(t)\le 0$ in $t\in[0,\pi]$, we have $|I_3|\le M[P_r(\delta)-P_r(\pi)]\le 4Mr(1-r)/(1+r^2-2r\cos\delta)\le M(1-r)/(r\sin^2\delta/2)$ $\le M(1-r)/[r(\delta/\pi)^2]=\pi^2M(1-r)/r\delta^2=\pi^2M(1-r)/r(1-r)^{4/\alpha}=\pi^2Mr^{-1}$ $\times(1-r)^{1-4/\alpha}\to 0$ $(r\to1)$, similarly $\lim\limits_{r\to1}I_1=0$. In the evaluation of I_2 we shall set for every $\varepsilon>0$, $A(\varepsilon)=E[t: |f(t)|<\varepsilon|t|]$,[5] $B(\varepsilon)=E[t: |f(t)|$ $\ge\varepsilon|t|]$, $p_\varepsilon(\delta)=\text{mes.}([0,\delta]\cdot B(\varepsilon))/\delta^\alpha$ and $I_2=\int_{-\delta}^{\delta}f(t)\dfrac{\partial}{\partial t}P_r(t)dt=\int_{[-\delta,\delta]\cdot A(\varepsilon)}$ $+\int_{[-\delta,\delta]\cdot B(\varepsilon)}=I_{2,1}+I_{2,2}$. First, as $I_{2,1}=\int_{[-\delta,\delta]\cdot A(\varepsilon)}\dfrac{f(t)}{t}t\dfrac{\partial}{\partial t}P_r(t)dt$ we have $|I_{2,1}|\le\varepsilon\int_{-\pi}^{\pi}\left|t\dfrac{\partial}{\partial t}P_r(t)\right|dt=\varepsilon K$, where K is a constant not depending on r. Secondly, setting $I_{2,2}=\int_{[0,\delta]\cdot B(\varepsilon)}+\int_{[-\delta,0]\cdot B(\varepsilon)}=I_{2,2}^{(1)}+I_{2,2}^{(2)}$, we have $|I_{2,2}^{(1)}|$ $\le M\int_{[0,\delta]\cdot B(\varepsilon)}\left|\dfrac{\partial}{\partial t}P_r(t)\right|dt\le M\int_{[0,\delta]\cdot B(\varepsilon)}K'/(1-r)^2\,dt=MK'\,\text{mes.}([0,\delta]\cdot B(\varepsilon))$ $(1-r)^{-2}=MK'\,\text{mes.}([0,\delta]\cdot B(\varepsilon))/\delta^\alpha=MK'p_\varepsilon(\delta)$, where K' is an absolute constant. The hypothesis that $f_{ap}^{[\alpha]}(0)=0$ implies $\lim\limits_{r\to1}p_\varepsilon(\delta)=0$. Therefore we have $\lim\limits_{r\to1}I_{2,2}^{(1)}=0$, in the same way, $\lim\limits_{r\to1}I_{2,2}^{(2)}=0$ and this completes the proof.

5) The notation being that $E[t: (\ \ )]$ is the set of all t such that $(\ \ )$.