26. Note on Fractional Powers of Linear Operators

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In the preceding paper by K. Yosida,[1] it is shown that the fractional power A^α, $0<\alpha<1$, of a linear operator A in a Banach space X can be constructed whenever $-A$ is the infinitesimal generator of a strongly continuous, bounded semi-group $\{\exp(-tA)\}$, and that $-A^\alpha$ also generates a semi-group $\{\exp(-tA^\alpha)\}$ which has an *analytic* extension in a sector containing the positive t-axis. In the present paper we shall give another proof of these results, together with some generalizations.

We consider linear operators in X which are not necessarily infinitesimal generators of semi-groups. For brevity we shall say that A is of type (ω, M)[2] if

i) A is densely defined[3] and closed, and

ii) the resolvent set of $-A$ contains the open sector $|\arg\lambda|<\pi-\omega$, $0<\omega<\pi$, and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$; in particular

$$\lambda\|(\lambda+A)^{-1}\|\leq M, \qquad \lambda>0. \tag{1}$$

As is well known, $-A$ is the infinitesimal generator of a strongly continuous contraction semi-group if and only if A is of type $(\pi/2, 1)$.

Theorem 1.[4] *Let A be of type (ω, M) with $\omega<\pi/2$. Then $-A$ is the infinitesimal generator of a semi-group $\{T_t\}_{t\geq 0}=\{\exp(-tA)\}$ with the following properties.*

a) *T_t has an analytic extension for $|\arg t|<\dfrac{\pi}{2}-\omega$.*

b) *In each smaller sector $|\arg t|<\dfrac{\pi}{2}-\omega-\varepsilon$, $\varepsilon>0$, T_t and $t\,dT_t/dt$*

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1) K. Yosida: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad., **36**, 86–89 (1960). For convenience we deviate from his notation in denoting by $-A$ instead of A the infinitesimal generator of a semi-group. The author is indebted to Professor Yosida for his suggestion to this problem.

2) A similar class of operators is considered by M. A. Krasnosel'skii and P. E. Sobolevskii, Doklady Acad. Nauk USSR, **129**, 499 (1959) and other Russian authors cited in this paper. But it appears that the semi-groups generated by $-A^\alpha$ are not considered by them.

3) This is a consequence of ii) if X is locally sequentially weakly compact, see T. Kato: Proc. Japan Acad., **35**, 467 (1959).

4) In case $M=1$, this theorem is contained in K. Yosida: Proc. Japan Acad., **34**, 337 (1958). Cf. also E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, Am. Math. Soc. Colloq. Publ., Vol. 31, Theorems 12.8.1 and 17.5.1 (1957).

are uniformly bounded and T_t coverges strongly to 1 (*=identity*) *for* $t\to 0$.

Proof. T_t is given by the Laplace transformation

$$T_t=\exp(-tA)=\frac{1}{2\pi i}\int_L e^{\lambda t}(\lambda+A)^{-1}d\lambda, \tag{2}$$

where the integration path L runs in the sector $|\arg\lambda|<\pi-\omega$ from $\infty e^{-i\theta_1}$ to $\infty e^{i\theta_2}$ with $\frac{\pi}{2}<\theta_1, \theta_2<\pi-\omega$. The assertions are easily proved by choosing θ_1, θ_2 appropriately. In proving b) it is convenient to introduce the new integration variable $\zeta=t\lambda$.

Theorem 2. *Let A be of type (ω, M). The fractional power A^α, $0<\alpha<1$, can be defined through*[5]

$$(\lambda+A^\alpha)^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}(\mu+A)^{-1}d\mu \tag{3}$$

which is valid for λ on and near the positive real axis. The operator A^α is of type $(\alpha\omega, M)$. If $\alpha\omega<\pi/2$, $-A^\alpha$ is the infinitesimal generator of an analytic semi-group $\{T_{t,\alpha}\}$ of the type described in Theorem 1.

Remark. If $-A$ is the infinitesimal generator of a strongly continuous, bounded semi-group, $\{T_{t,\alpha}\}$ is defined for $0<\alpha<1$ and also a bounded semi-group (for real t). A^α and $T_{t,\alpha}$ coincide with the corresponding operators defined by Yosida.[1]

Proof. I. For any λ on or near the positive real axis, the integral in (3) is absolutely convergent by ii). Let us denote by $R(\lambda)$ the bounded linear operator thus defined by the right member of (3). $R(\lambda)$ satisfies the resolvent equation

$$R(\lambda)-R(\lambda')=-(\lambda-\lambda')R(\lambda)R(\lambda'). \tag{4}$$

This could be verified by a direct calculation, but the following consideration seems to be simpler. For the moment assume that the origin 0 belongs to the resolvent set of A. Then it is easily seen that $R(\lambda)$ is given by the complex integral

$$R(\lambda)=\frac{1}{2\pi i}\int_C(\lambda+z^\alpha)^{-1}(A-z)^{-1}dz \tag{5}$$

where $\lambda>0$ and the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega<\theta<\pi$, avoiding the negative real axis and 0; (3) is obtained from (5) by deforming C to the upper and lower banks of the negative real axis. The absolute convergence of the integral in (5) also follows from ii). Since (5) is a kind of Dunford integral, it is easy to see that $R(\lambda)$ satisfies (4). In the general case, we replace A by $A+\varepsilon$ with $\varepsilon>0$ and let $\varepsilon\to 0$ afterwards. Since the right member of (3) with A

5) If A^{-1} is bounded, (3) is true even for $\lambda=0$ and coincides with the operator $A^{-\alpha}$ defined in 2).

replaced by $A+\varepsilon$ converges for $\varepsilon\to 0$ strongly to $R(\lambda)$, it follows that $R(\lambda)$ satisfies (4).

II. Hence $R(\lambda)$ can be expressed in the form $(\lambda+A^\alpha)^{-1}$ with a closed linear operator A^α, provided that $R(\lambda)$ has the (common) null space $\{0\}$. But this follows from the strong convergence $\lambda R(\lambda)\to 1$, $\lambda\to+\infty$, which can be deduced from (3) and the fact that $\lambda(\lambda+A)^{-1}\to 1$. At the same time this shows that A^α is densely defined.

III. It is easily seen from (3) that $R(\lambda)$ is defined and analytic in the sector $|\arg\lambda|<(1-\alpha)\pi$. But it can further be continued analytically to the larger sector $|\arg\lambda|<\pi-\alpha\omega$. To see this it suffices to regard the integral in (3) as a complex integral and shift the integration path to the ray $\arg\mu=\pm(\pi-\omega-\varepsilon)$ with a small $\varepsilon>0$. A simple homogeneity consideration shows also that ii) is satisfied for A^α with ω replaced by $\alpha\omega$. In particular for $\lambda>0$, (3) and (1) give

$$\|(\lambda+A^\alpha)^{-1}\|\le\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}\cdot\frac{M}{\mu}d\mu=\frac{M}{\lambda}. \tag{6}$$

This completes the proof that A^α is of type $(\alpha\omega, M)$. The last statement of Theorem 2 then follows from Theorem 1.

IV. It remains to show that $\{T_t\}$ coincides with the semi-group constructed by Yosida. To this end we first consider the special case in which $-(A-\varepsilon)$, for some $\varepsilon>0$, is the infinitesimal generator of a bounded semi-group, so that the half-plane $\mathrm{Re}\, z<\varepsilon$ belongs to the resolvent set of A. Since $\omega=\pi/2$, the path C of (5) can be chosen in such a way that we have $\mathrm{Re}\, z<\varepsilon$ and $|\arg z^\alpha|\le\phi<\pi/2$ for $z\in C$. Then (5) is valid for all λ with $|\arg\lambda|\le\pi-\phi(>\pi/2)$. Take the path L in (2) in such a way that this condition is satisfied for all $\lambda\in L$. Then we have from (2) and (5) (note that $(\lambda+A^\alpha)^{-1}=R(\lambda)$)

$$\begin{aligned} T_{t,\alpha}=\exp(-tA^\alpha)&=\Big(\frac{1}{2\pi i}\Big)^2\int_L e^{\lambda t}d\lambda\int_C(\lambda+z^\alpha)^{-1}(A-z)^{-1}dz\\ &=\frac{1}{2\pi i}\int_C e^{-tz^\alpha}(A-z)^{-1}dz\\ &=\frac{1}{2\pi i}\int_C e^{-tz^\alpha}\,dz\int_0^\infty e^{\tau z}T_\tau d\tau\qquad (T_t=\exp(-tA))\\ &=\frac{1}{2\pi i}\int_0^\infty T_\tau d\tau\int_C e^{\tau z-tz^\alpha}dz. \end{aligned} \tag{7}$$

This shows[6] that our $\{T_{t,\alpha}\}$ coincides with the semi-group defined by Yosida. The general case can be dealt with by replacing A by $A+\varepsilon$ and letting $\varepsilon\to 0$; it suffices to note that[7] the strong convergence $[\lambda+(A+\varepsilon)^\alpha]^{-1}\to(\lambda+A^\alpha)^{-1}$, $\varepsilon\to 0$, $\lambda>0$, already proved implies the strong convergence $\exp[-t(A+\varepsilon)^\alpha]\to\exp(-tA^\alpha)$, $t>0$.

6) See Eqs. (10) and (16) of Yosida.[1]

7) See e.g. H. F. Trotter: Pacific J. Math., 8, 887 (1958).