8. On Transformation of Manifolds

By Joseph WEIER

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1960)

Let $m>n>r\geq 1$ be integers, suppose M is an m-dimensional and N an n-dimensional oriented closed polyhedral manifold, let S be the simplicial image of an oriented r-sphere situated in N, and $f: M\rightarrow N$ a continuous mapping. Then one may suppose that $f^{-1}(S)$ is a finite polyhedron R in M satisfying

$$\dim R=m-n+r.$$

Let $A_1, A_2,\cdots$ be the $(m-n+r)$-simplexes of a simplicial decomposition of R, moreover A one of the A_i, and A^* an orientation of A. The simplexes used here are open and rectilinear. If a is a point in A, one can suppose S is smooth in a neighborhood of the point $b=f(a)$. Let B be an r-simplex with $b\in B\subset S$. Define C to be an $(n-r)$-simplex in M perpendicular to A, and D an $(n-r)$-simplex in N perpendicular with respect to B such that $A\cap C=a$, $B\cap D=b$, $R\cap \bar{C}=a$, and $S\cap \bar{D}=b$. For every point $p\in\partial C$, let $\varphi(p)$ denote the vertical projection of $f(p)$ on D parallel to B. Then $\varphi(\partial C)\subset D-b$. For $p\in\partial C$, let $\varphi'(p)$ be the vertical projection of $\varphi(p)$ on ∂D out of b. By C^* we denote an orientation of C such that (A^*, C^*) gives the positive orientation of M, by B^* the orientation of B induced by S, and by D^* an orientation of D such that (B^*, D^*) furnishes the positive orientation of N. Let $\beta(A^*)$ be the Brouwer degree of the map $\varphi': \partial B^*\rightarrow\partial D^*$.

Let a_k be an orientation of A_k and β_k the number $\beta(a_k)$. Then $\sum\beta_k a_k$ represents a finite $(m-r+r)$-cycle that we will denote by $\sigma_f(S)$ as well. If the continuous r-sphere S' is homotopic to S within N, then

$$\sigma_f(S)\sim\sigma_f(S').$$

Let $\pi_r(N)$ be the r-dimensional Hurewicz group of N. Define h to be the homotopy class of S, and $\zeta(h)$ to be the homology class of $\sigma_f(S)$. Then the mapping $\zeta:\pi_r(N)\rightarrow H_{m-n+r}(M)$, where $H_i(M)$ means the i-dimensional integral Betti group of M, is a homomorphism. Of course, the latter is related to known inverse homomorphisms. But for the following it is important to have an exact *geometric realization* of these homomorphisms; a problem to which already Whitney [4] has hinted.

Now suppose $r=2n-m-1\geq 2$, and let $\pi_r^\zeta(N)$ be the kernel of the homomorphism ζ, moreover h_r^ζ an element of $\pi_r^\zeta(N)$, and Q an oriented continuous sphere of h_r^ζ. One may suppose $f^{-1}(Q)$ is an $(m-n+r)$-polyhedron in M. Denote the cycle $\sigma_f(Q)$ by z as well. Evidently,

$\dim z = n-1$. By $\zeta\pi_r^\xi(N)=0$,

$$z \sim 0.$$

Define two n-chains y and Y' of M to belong to the same *equivalence class* with respect to z if

$$\partial y = \partial y' = z \quad \text{and} \quad y' - y \sim 0.$$

Let $Y_i(z)$, $i=1,2,\cdots$, be the equivalence classes thus obtained, and suppose y_i is a chain of $Y_i(z)$. Then, for all pairs (i, j),

$$y_i - y_j$$

is an n-cycle y_{ij} in M with integral coefficients. Denote the degree of the mapping $f: y_{ij} \to N$ by $\beta_{ij}(z)$. Then the system of the numbers

(1) $$\beta_{ij}(z), \quad i=1,2,\cdots, \quad j=1,2,\cdots,$$

is uniquely determined in the following sense:

If one represents h_r^ξ, instead of by Q, by another sphere, if z' denotes the cycle corresponding to z, and if

$$\beta'_{ij}(z'), \quad i=1,2,\cdots, \quad j=1,2,\cdots,$$

are the numbers that correspond to the $\beta_{ij}(z)$, then one can assign a pair $\varphi(i,j)$ to every (i,j) satisfying $\beta_{ij}(z) \neq 0$ in such a way that, firstly,

$$\beta_{ij}(z) = \beta'_{\varphi(i,j)}(z')$$

and that, secondly, the following holds: corresponding to each (k,l) with $\beta'_{kl}(z') \neq 0$ there exists just one (i,j) with $\varphi(i,j)=(k,l)$.

Thus, while in the classical case each transformation of an oriented closed manifold in a second one of the same dimension possesses only one degree, the pairs (m, n) with

$$m \leq 2n-3$$

furnish the system (1) that in general consists of an infinite number of degrees. By the way, $n \geq 3$ since we had supposed above that $m > n$. *Apart from permutations and zeros, the system* (1) *is invariant under deformation of f.*

Besides the pairs (m, n) with $n < m \leq 2n-3$ above discussed, we will regard still another series of pairs: the positive integers m, n satisfying

$$2n \leq m \leq 3n-2.$$

Let the meaning of M, N, and $f: M \to B$ be the same as before. Let r be the number $r = 3n-m-1$. Suppose the cycle z and the equivalence classes $y_1, y_2, \cdots$ to be defined as before. Evidently,

$$\dim z = 2n-1 \quad \text{and} \quad \dim y_i = 2n.$$

In every neighborhood of f, there exists a map f' homotopic to f such that the set consisting of all points $p \in M$ with $f(p) = f'(p)$ is a finite $(m-n)$-polyhedron W. Now let w_k be the oriented $(m-n)$-simplexes of a simplicial decomposition of W, and define γ_k to the degree of w_k with respect to (f, f'). Then $\sum \gamma_k w_k$ is an $(m-n)$-cycle, w, with integral coefficients. For all (i, j), let x_{ij} be the intersection cycle of y_{ij}

and w. Then $\dim x_{ij} = \dim y_{ij} + \dim w - \dim M = n$.

Let $\gamma_{ij}(z)$ be the degree of the mapping $f: x_{ij} \to N$. Then the system of the numbers

$$\gamma_{ij}(z), \quad i=1,2,\cdots, \quad j=1,2,\cdots, \tag{2}$$

is, apart from permutations and zeros, uniquely determined by the homotopy classes of Q and f.

We will conclude by recalling three recent papers [1-3] on the degree. In addition, we should remark that each of the degrees β_{ij} and γ_{ij} is decomposable in Nielsen components β_{ijk} and γ_{ijk} with

$$\sum_k \beta_{ijk} = \beta_{ij} \quad \text{and} \quad \sum_k \gamma_{ijk} = \gamma_{ij}$$

that, on their part, are invariant under homotopies. The de Rham isomorphism theorem furnishes integral expressions for the β_{ij} and γ_{ij}.

References

[1] Eilenberg, S., and Steenrod, N.,: Foundations of algebraic topology, Princeton Math., ser. 15, 298-322 (1952).

[2] Fuller, F. B.,: A relation between degree and linking numbers, Princeton Math., ser. 12, 258-262 (1956).

[3] Rothe, E.,: Ueber den Abbildungsgrad bei Abbildungen von Kugeln des Hilbertraumes, Zur Theorie der topologischen Ordnung und der Vektorfelder in Banachschen Räumen, Compositio Math., **5**, 166-176, 177-197 (1938).

[4] Whitney, H.,: Geometric methods in cohomology theory, Proc. Nat. Acad. Sci. U. S. A., **33**, 7-9 (1947).