25. On Pseudo-compact Spaces

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Notations. Let S be a topological space. Let Z be the family of all sequences $\{f_n(x)\}_{n=1,2,\dots}$, where f_n are (finite real) continuous functions on S such that $f_n(x)\to 0$ for each $x\in S$. Let Z_0 be the family of all bounded sequences $\{f_n\}\in Z$; let N (resp. E, resp. U) be the family of all non-increasing (resp. equi-continuous, resp. uniformly convergent) sequences $\{f_n\}\in Z$. Further we put $N_0=N\frown Z_0$, $E_0=E\frown Z_0$.

If b is a real number, we write $b_+=\max(b, 0)$.

Lemma 1. $U\smile N\subset E$.

Proof. If $\{f_n\}\in U\smile N$, $x\in S$, $\varepsilon=2\eta>0$, then there exist an index p and a neighbourhood V of x such that $|f_n(y)|<\eta$ for each $n>p$ and each $y\in V$. Further we can find a neighbourhood W of x such that $|f_n(x)-f_n(y)|<\varepsilon$ for $n=1,\cdots,p$ and for each $y\in W$. Obviously $|f_n(x)-f_n(y)|<\varepsilon$ for each n and each $y\in V\frown W$.

Lemma 2. *If* $\{f_n\}\in E$, *then the function* $f(x)=\sum_{n=1}^{\infty}(|f_n(x)|-\varepsilon)_+$ *is continuous for each* $\varepsilon>0$.

Proof. Suppose that $x\in S$ and that $\varepsilon=2\eta>0$. There exist an index p and a neighbourhood V of x such that $|f_n(x)|<\eta$ for each $n>p$ and that $|f_n(x)-f_n(y)|<\eta$ for each $y\in V$ and each n. Now, if $n>p$ and if $y\in V$, we have $|f_n(y)|<\varepsilon$, whence $(|f_n(y)|-\varepsilon)_+=0$. It follows that $f(y)=\sum_{n=1}^{p}(|f_n(y)|-\varepsilon)_+$ for each $y\in V$, which completes the proof.

Lemma 3. *If S is pseudo-compact, then* $N\subset U$.

Proof. If $\{f_n\}\in N$ and if $\varepsilon=2\eta>0$, then, by Lemmas 1 and 2, the function $f(x)=\sum_{n=1}^{\infty}(f_n(x)-\eta)_+$ is continuous. Since S is pseudo-compact, there exists a number A such that $f(x)<A$ for each $x\in S$. If $f_n(x)\geq\varepsilon$, then $(f_k(x)-\eta)_+\geq\eta$ for $k=1,2,\cdots,n$, so that $n\eta\leq f(x)<A$, $n<A\eta^{-1}$. We see that $f_n(x)<\varepsilon$ for each $n\geq A\eta^{-1}$ and for each $x\in S$.

Lemma 4. *If S is pseudo-compact, then* $E\subset U$.

Proof. Let $\{f_n\}$ be a sequence of E and let $\varepsilon=2\eta$ be a positive number. Lemma 2 implies that the functions $g_n(x)=\sum_{k=n}^{\infty}(|f_k(x)|-\eta)_+$ are continuous; obviously $\{g_n\}\in N$ and so, by Lemma 3, $\{g_n\}\in U$. There exists an index p such that $g_p(x)<\eta$ for each $x\in S$. If $n\geq p$, then we have $|f_n(x)|\leq(|f_n(x)|-\eta)_++\eta\leq g_p(x)+\eta<2\eta=\varepsilon$ for each $x\in S$, which proves the lemma.

Lemma 5. *If* $N_0\subset U$, *then S is pseudo-compact.*

Proof. Let f be an unbounded continuous function on S; put $f_n(x)=\operatorname{arctg}(n^{-1}|f(x)|)$. Then $\{f_n\}\in N_0-U$.

Theorem. If some of the families E, E_0, N, N_0 is contained in U, then S is pseudo-compact. If, conversely, S is pseudo-compact, then $U=E=E_0$, $U\ni N=N_0$.

Proof. Lemma 1 implies that $N\subset E$; now it is obvious that $N_0\subset N$, $N_0\subset E_0\subset E$. If some of the families E, E_0, N, N_0 is contained in U, then we have $N_0\subset U$ and, by Lemma 5, S is pseudo-compact.

Now let S be pseudo-compact. From Lemmas 1 and 4 we see that $E=U$, so that $E=E_0$. Lemma 3 implies that $N\subset U$; obviously $N=N_0$ and the proof is complete.

Remark. This theorem is a slight generalization of Theorem 3 of [1] and of Theorem 3 of [2].

References

[1] K. Iséki: A characterisation of pseudo-compact spaces, Proc. Japan Acad., **33**, 320–322 (1957).

[2] K. Iséki: Pseudo-compactness and strictly continuous convergence, Proc. Japan Acad., **33**, 424–428 (1957).