132. Some Notes on Cesàro Summation

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In this paper we shall establish two lemmas concerning the Cesàro summability of Fourier series. Of these, Theorem 1 is closely related to the result of Chandrasekharan and Szász [2, Theorem 5]. And Theorem 2 is concerned with the estimation of the principal part of Fejér kernels.

1. THEOREM 1. If $\varphi(t)\in L$ in $0\leqq t\leqq t_0$, and $r>0$, $\delta>0$, and q be arbitrary, then

$$\Phi_r(t)\equiv\frac{1}{\Gamma(r)}\int_0^t(t-u)^{r-1}\varphi(u)du=o(t^q)\qquad(t\to 0)\tag{1.1}$$

is equivalent to

$$\Phi_r^\delta(t)\equiv\frac{1}{\Gamma(r)}\int_0^t(t-u)^{r-1}u^\delta\varphi(u)du=o(t^{q+\delta})\qquad(t\to 0).\tag{1.2}$$

Letting

$$\varphi_r^\delta(t)=\frac{\Gamma(r+\delta+1)}{\Gamma(\delta+1)}t^{-(r+\delta)}\Phi_r^\delta(t)\qquad(\delta\geqq 0),$$

and $\varphi_r(t)=\varphi_r^0(t)$, we have the following

COROLLARY 1. Let $\varphi(t)\in L$ in $(0, t_0)$, and $r>0$, $\delta>0$, and q be arbitrary. Then

$$\varphi_r(t)=s+o(t^{q-r})\qquad(t\to 0)$$

is equivalent to

$$\varphi_r^\delta(t)=s+o(t^{q-r})\qquad(t\to 0),$$

where s is a constant independent of t.

Concerning this corollary, cf. loc. cit. [2].

We need two lemmas:

LEMMA 1. Theorem 1 holds when $\delta=k$, where k is a positive integer.

This is Lemma 3 in the paper [3], but for the sake of completeness we prove it. We first consider the case $k=1$. Observe now that

$$\Phi_r^1(t)=t\Phi_r(t)-r\Phi_{r+1}(t),\tag{1.3}$$

and that necessarily, since $r>0$,

$$\Phi_{r+1}(t)=o(t^r).\tag{1.4}$$

If $q>-1$, then (1.1) implies

$$\Phi_{r+1}(t)=o(t^{q+1}),\tag{1.5}$$

and then by (1.3),

$$\Phi_r^1(t)=o(t^{q+1}),\tag{1.6}$$

which follows from (1.1) still when $q\leqq -1$, by (1.3) and (1.4).

Inversely, (1.6) is written as, by (1.3),

$$\frac{\Phi_r(t)}{t^r}-r\frac{\Phi_{r+1}(t)}{t^{r+1}}=o(t^{q-r}),$$

i.e.

$$\frac{d}{dt}[t^{-r}\Phi_{r+1}(t)]=o(t^{q-r}).$$

If $q-r>-1$, integrating both sides from zero to t, we have $t^{-r}\Phi_{r+1}(t)=o(t^{q-r+1})$, by (1.4), which is equivalent to (1.5). And, (1.5) holds still when $q-r\leqq-1$ again by (1.4). Consequently, (1.6) implies (1.1) by (1.3).

We have thus the lemma when $k=1$. In the general case $k>1$, replacing $\varphi(u)$ by $u\varphi(u)$, $u^2\varphi(u)$, $\cdots$, successively it is proved by induction.

LEMMA 2. If $\varphi(t)\in L$ in $(0,x)$ and $0<y<x$, $0<r\leqq1$, then

$$\left|\frac{1}{\Gamma(r)}\int_0^y(x-t)^{r-1}\varphi(t)dt\right|\leqq\max_{0\leqq u\leqq x}|\Phi_r(u)|.$$

This is due to Riesz [1].

PROOF of THEOREM 1. (I) The case $0<r\leqq1$, $r\leqq q$. By the second mean-value theorem

$$\begin{aligned}\Phi_r^{\delta+\eta}(t)&=\frac{1}{\Gamma(r)}\int_0^t(t-u)^{r-1}u^{\delta+\eta}\varphi(u)du\\&=\frac{t^\delta}{\Gamma(r)}\int_\xi^t(t-u)^{r-1}u^\eta\varphi(u)du \qquad (0<\xi<t)\\&=t^\delta\Phi_r^\eta(t)-\frac{t^\delta}{\Gamma(r)}\int_0^\xi(t-u)^{r-1}u^\eta\varphi(u)du,\end{aligned}$$

where $\delta>0$ and $\eta\geqq0$. So, by Lemma 2, we have

$$|\Phi_r^{\delta+\eta}(t)|\leqq2t^\delta\cdot\max_{0\leqq u\leqq t}|\Phi_r^\eta(u)|. \tag{1.7}$$

Hence, if $\Phi_r(t)=o(t^q)$, then (1.7) with $\eta=0$ yields $\Phi_r^\delta(t)=o(t^{\delta+q})$, since $q>0$. Inversely, if $\Phi_r^\eta(t)=o(t^{q+\eta})$, then (1.7) with $\delta=[\eta]+1-\eta$ yields

$$\Phi_r^{[\eta]+1}(t)=t^{[\eta]+1-\eta}\cdot o(t^{\eta+q})=o(t^{[\eta]+1+q}),$$

which implies $\Phi_r(t)=o(t^q)$ by Lemma 1, since $[\eta]+1$ is integral. Hence, we get the present case.

(II) The case $1<r\leqq q$. We have the identities

$$\begin{aligned}\Phi_r^\delta(t)=&\frac{(r-1)t}{\Gamma(r)}\int_0^t(t-u)^{r-2}u^{\delta-1}\Phi_1(u)du\\&-\frac{r-1+\delta}{\Gamma(r)}\int_0^t(t-u)^{r-1}u^{\delta-1}\Phi_1(u)du,\end{aligned} \tag{1.8}$$

$$\Phi_r^\delta(t)=\frac{t^{r+\delta}}{\Gamma(r)}\frac{d}{dt}\left(\frac{1}{t^{r-1+\delta}}\int_0^t(t-u)^{r-1}u^{\delta-1}\Phi_1(u)du\right). \tag{1.9}$$

And, (1.1) is equivalent to, since $r>1$,

$$(1.1)' \qquad \int_0^t (t-u)^{r-2}\Phi_1(u)du = o(t^q).$$

Suppose now that the theorem is true when r is replaced by $r-1$. Then, (1.1)' is equivalent to

$$(1.10) \qquad \int_0^t (t-u)^{r-2}u^{\delta-1}\Phi_1(u)du = o(t^{q+\delta-1}), \quad \delta>1,$$

which clearly implies

$$(1.11) \qquad \int_0^t (t-u)^{r-1}u^{\delta-1}\Phi_1(u)du = o(t^{q+\delta}).$$

Substituting this and (1.10) into (1.8), we have

$$\Phi_r^\delta(t) = o(t^{q+\delta}), \quad \delta>1,$$

and then $\Phi_r^{\delta-1}(t) = o(t^{q+\delta-1})$ by Lemma 1.

Inversely, if (1.2) holds, i.e. $\Phi_r^\delta(t)=o(t^{q+\delta})$ then by integrating, we have (1.11) for $\delta>0$ from (1.9). (1.11) and (1.2) imply (1.10) by (1.8), and then (1.1)' by the above assumption. We thus get the present case by induction.

(III) General case $r>0$, q arbitrary. We put

$$\psi(u) = u^k\varphi(u),$$

where k is a positive integer such that $k+q>r$, and define

$$\Psi_r^\delta(t) = \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}u^\delta\psi(u)du \qquad (\delta\geqq 0),$$

and $\Psi_r(t)=\Psi_r^0(t)$. Then

$$(1.12) \qquad \Psi_r(t)=\Phi_r^k(t), \quad \Psi_r^\delta(t)=\Phi_r^{k+\delta}(t) \qquad (\delta>0).$$

By the preceding result, we see that

$$\Psi_r(t)=o(t^{k+q}) \Longleftrightarrow \Psi_r^\delta(t)=o(t^{k+q+\delta}),$$

since $k+q>r$. This is the same thing as, by (1.12),

$$\Phi_r^k(t)=o(t^{k+q}) \Longleftrightarrow \Phi_r^{k+\delta}(t)=o(t^{k+q+\delta}),$$

whence follows, by Lemma 1,

$$\Phi_r(t)=o(t^q) \Longleftrightarrow \Phi_r^\delta(t)=o(t^{q+\delta}).$$

This proves the theorem completely.

2. THEOREM 2.1. If $0<\delta<1$, $-1<\beta$, $1\leqq k$ and $0<u<\pi$, then we have

$$\begin{aligned} F(u,k) &\equiv \int_u^\pi (t-u)^{\delta-1}\left(2\sin\frac{1}{2}t\right)^{-\beta} e^{ikt}dt, \quad i=\sqrt{-1}, \\ &= \frac{\Gamma(\delta)}{k^\delta}\left(2\sin\frac{1}{2}u\right)^{-\beta} e^{i(ku+\delta\pi/2)} + 2^{-\beta}(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{ik} \\ &\quad + O\left(\frac{(\pi-u)^{\delta-2}}{k^2u^\beta}\right) + O\left(\frac{1}{k^{\delta+1}u^{\beta+1}}\right), \end{aligned}$$

where O's are independent of u and k.

PROOF.

$$(2.1)\qquad \begin{aligned} F(u,k) &= \left(2\sin\frac{1}{2}u\right)^{-\beta}\int_u^\pi (t-u)^{\delta-1}e^{ikt}dt \\ &\quad + \int_u^\pi (t-u)^\delta m(t,u)e^{ikt}dt = \left(2\sin\frac{1}{2}u\right)^{-\beta} I + J, \end{aligned}$$

where $0<u<\pi$, and

$$(2.2)\qquad m(t,u)=\frac{1}{t-u}\left[\left(2\sin\frac{1}{2}t\right)^{-\beta}-\left(2\sin\frac{1}{2}u\right)^{-\beta}\right].$$

Here, for the sake of convenience we denote

$$(2.3)\qquad m(u,u)=\lim_{t\to u} m(t,u)=\frac{d}{du}\left(2\sin\frac{1}{2}u\right)^{-\beta}.$$

By the mean-value theorem,

$$m(t,u)=m(u,u)\,|_{u=u_1} \qquad (u<u_1<t).$$

And, clearly

$$(2.4)\qquad \frac{\partial}{\partial t}m(t,u)=\frac{1}{t-u}[m(t,t)-m(t,u)].$$

From these relations we see that $m(t, u)$ conserves a constant sign for $0<u<t\leqq\pi$, and increases with $1/t$ in absolute value, and that

$$(2.5)\qquad |m(t,u)|<|m(u,u)|<\frac{K}{u^{\beta+1}},$$

where and in the sequel K denotes an absolute constant, and it may vary from one occurrence to another. Now,

$$(2.6)\qquad I=\int_u^\pi (t-u)^{\delta-1}e^{ikt}dt=\int_u^\infty - \int_\pi^\infty .$$

And

$$\begin{aligned} \int_u^\infty &= \int_u^\infty (t-u)^{\delta-1}e^{ikt}dt \\ &= e^{iku}\int_0^\infty x^{\delta-1}e^{ikx}dx \qquad (t-u=x) \\ &= e^{iku}\cdot\frac{\Gamma(\delta)}{k^\delta}e^{i\delta\pi/2}, \end{aligned}$$

by a well-known classical formula, cf. Zygmund [4, p. 224].

$$\begin{aligned} \int_\pi^\infty &= \int_\pi^\infty (t-u)^{\delta-1}e^{ikt}dt \\ &= \left[(t-u)^{\delta-1}\frac{e^{ikt}}{ik}\right]_{t=\pi}^\infty -(\delta-1)\int_\pi^\infty (t-u)^{\delta-2}\frac{e^{ikt}}{ik}dt \\ &= -(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{ik}-(\delta-1)I_1, \end{aligned}$$

and clearly $I_1=O((\pi-u)^{\delta-2}/k^2)$. Hence, from (2.6) we get

$$(2.7)\qquad I=\frac{\Gamma(\delta)}{k^\delta}e^{i(ku+\delta\pi/2)}+(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{ik}+O\left(\frac{(\pi-u)^{\delta-2}}{k^2}\right).$$

Next, integrating by parts and using (2.4),

$$
\begin{aligned}
J&=\int_u^\pi (t-u)^\delta m(t,u)e^{ikt}dt\\
&=\left[(t-u)^\delta m(t,u)\frac{e^{ikt}}{ik}\right]_{t=u}^{\pi}-(\delta-1)\int_u^\pi (t-u)^{\delta-1}m(t,u)\frac{e^{ikt}}{ik}dt\\
&\qquad\qquad -\int_u^\pi (t-u)^{\delta-1}m(t,t)\frac{e^{ikt}}{ik}dt \qquad (2.8)\\
&=(\pi-u)^\delta m(\pi,u)\frac{e^{ik\pi}}{ik}-(\delta-1)J_1-J_2.
\end{aligned}
$$

By the monotonity of $m(t,u)$, and (2.5),

$$|J_1|<K\frac{|m(u,u)|}{k^{\delta+1}}<\frac{K_1}{k^{\delta+1}u^{\beta+1}}.$$

It is analogous to J_2. Hence, from (2.8) and (2.2),

$$J=(\pi-u)^{\delta-1}\left[2^{-\beta}-\left(2\sin\frac{1}{2}u\right)^{-\beta}\right]\frac{e^{ik\pi}}{ik}+O\left(\frac{1}{k^{\delta+1}u^{\beta+1}}\right). \tag{2.9}$$

Substituting (2.7) and (2.9) into (2.1) we get the theorem.

Theorem 2.1 may be improved more precisely as follows:

THEOREM 2. If $0<\delta<1$, $-1<\beta$, $1\leqq k$ and $0<u<\pi$, then

$$
\begin{aligned}
&\int_u^\pi (t-u)^{\delta-1}\left(2\sin\frac{1}{2}t\right)^{-\beta}e^{ikt}dt\\
&=\frac{\Gamma(\delta)}{k^\delta}\left(2\sin\frac{1}{2}u\right)^{-\beta}e^{i(ku+\delta\pi/2)}\\
&\quad+\frac{\Gamma(\delta+1)}{k^{\delta+1}}\frac{d}{dt}\left(2\sin\frac{1}{2}u\right)^{-\beta}e^{i(ku+(\delta+1)\pi/2)}\\
&\quad+2^{-\beta}(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{ik}+(\delta-1)\cdot2^{-\beta}(\pi-u)^{\delta-2}\frac{e^{ik\pi}}{k^2}\\
&\quad+O\left(\frac{(\pi-u)^{\delta-3}}{k^3u^\beta}\right)+O\left(\frac{(\pi-u)^{\delta-2}}{k^3u^{\beta+1}}\right)+O\left(\frac{1}{k^{\delta+2}u^{\beta+2}}\right),
\end{aligned}
$$

where O's are independent of u and k.

PROOF. We use the notations in the preceding proof. Integrating by parts,

$$
\begin{aligned}
I_1&=\int_\pi^\infty (t-u)^{\delta-2}\frac{e^{ikt}}{ik}dt\\
&=(\pi-u)^{\delta-2}\frac{e^{ik\pi}}{k^2}+O\left(\frac{(\pi-u)^{\delta-3}}{k^3}\right).
\end{aligned}
$$

Applying Theorem 2.1 replaced $(2\sin(2^{-1}t))^{-\beta}$ by $m(t,u)$ to J_1, and observing that by (2.4) and (2.3)

$$\left|\frac{\partial}{\partial t}m(t,u)\right|<\frac{K}{u^{\beta+2}}$$

for $0<u<t\leqq\pi$, we have, as it is easily verified,

$$J_1=\int_u^\pi (t-u)^{\delta-1}m(t,u)\frac{e^{ikt}}{ik}dt$$
$$=\frac{\Gamma(\delta)}{k^\delta}\cdot\frac{m(u,u)}{ik}e^{i(ku+\delta\pi/2)}+m(\pi,u)(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{(ik)^2}+O(R),$$

where

$$R=\frac{(\pi-u)^{\delta-2}}{k^3u^{\beta+1}}+\frac{1}{k^{\delta+2}u^{\beta+2}}.$$

Similarly

$$J_2=\int_u^\pi (t-u)^{\delta-1}m(t,t)\frac{e^{ikt}}{ik}dt$$
$$=\frac{\Gamma(\delta)}{k^\delta}\cdot\frac{m(u,u)}{ik}e^{i(ku+\delta\pi/2)}+m(\pi,\pi)(\pi-u)^{\delta-1}\frac{e^{ik\pi}}{(ik)^2}+O(R),$$

where $m(\pi,\pi)=0$.

Substituting these relations into the expressions of I and J respectively, (2.1) yields the desired result.

References

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