97. A Characterisation of P-spaces

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The concept of P-spaces was introduced by L. Gillman and M. Henriksen [1]. Following their definitions, we shall define *P-point* and *P-space.*

A point x of a topological space X is called a *P-point* if every countable intersection of neighborhoods of x contains a neighborhood of x. X is called *P-space,* if every point of X is a P-point.

In my papers published in Proc. Japan Acad. (1957–1958), some topological spaces have been characterised by *the properties on sequences of continuous functions.* In this Note, we shall give a characterisation of P-space by a similar consideration. To do it, we suppose that a topological space X is completely regular. Then we have the following

Theorem. A completely regular space is P-space, if and only if the limit function of any convergent sequence of continuous functions is continuous.

Proof. Let $\{f_n(x)\}$ be a convergent sequence of continuous functions $f_n(x)$ on a completely regular P-space, and let $f(x)$ be its limit. To prove that $f(x)$ is continuous on X, for a point a of X and a given positive ε, we take the neighborhood $U_n(a)=\{x\,||f_n(a)-f_n(x)|<\varepsilon\}$ of a. Since X is P-space, $U=\bigcap_{n=1}^{\infty} U_n(a)$ is a neighborhood of a, and we have

$$|f_n(a)-f_n(x)|<\varepsilon \quad \text{for } x\in U \ (n=1,2,\cdots).$$

Therefore, for $n\to\infty$, we have

$$|f(a)-f(x)|\leq\varepsilon \quad \text{for } x\in U.$$

This shows that $f(x)$ is continuous at $a\in X$.

Conversely, suppose that there is a sequence of neighborhoods U_n of a such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of a. Since X is completely regular, for every n, we can find a continuous function $f_n(x)$ such that $0\leq f_n(x)\leq 1$ and

$$f_n(x)=\begin{cases} 0 & x\,\bar{\in}\, U_n, \\ 1 & x=a. \end{cases}$$

Let $g_n(x)=\text{Min}\{f_1(x), f_2(x),\cdots, f_n(x)\}$, then the sequence $\{g_n(x)\}$ is decreasing and $0\leq g_n(x)\leq 1$. Therefore $\{g_n(x)\}$ is convergent to a function $g(x)$ on X. Since the intersection $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of a, for a given neighborhood U of a, we can take a point b such that $U-\bigcap_{n=1}^{\infty} U_n\ni b$. Hence $b\,\bar{\in}\,\bigcap_{n=1}^{\infty} U_n$ and we can find a neighborhood U_{n_0} such that $b \bar{\in} U_{n_0}$. Therefore, we have $f_{n_0}(b)=0$, and consequently, $g_n(b)=0$ for $n \geq n$. This shows $g(b)=0$. From $g_n(a)=1$, we have $g(a)=0$. Therefore $|g(a)-g(b)|=1$, $a, b \in U$. It follows that $g(x)$ is not continuous at the point a, which is a contradiction. The proof is complete.

Added in the proof. We found that the same result has been obtained by N. Onuchic: On two properties of P-spaces, Portugaliae Math., 16, 37-39 (1957).

Reference

[1] L. Gillman and M. Henriksen: Concerning rings of continuous functions, Trans. Am. Math. Soc., **77**, 340-362 (1954).