74. On the Automorphisms of Some Simple Groups

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(Comm. by Z. SUETUNA, M.J.A., June 12, 1958)

In [2] C. Chevalley has constructed the simple groups of Lie types by a uniform method. These groups are isomorphic to the known classical groups if the simple Lie algebras are of the main four types (cf. [1] or [4]). Corresponding to the theory of automorphisms of classical groups studied by J. Dieudonné and L. K. Hua, we consider in this note the automorphisms of simple groups of Lie types of C. Chevalley in the special case where the ground field is algebraically closed and of characteristic 0. In this case, these simple groups are algebraic in the sense of Chevalley and every birational and biregular automorphism of these groups is inner except for the type (D_4), and if the group is of type (D_4), the factor group of the automorphism group by its normal subgroup of inner automorphisms is the cyclic group of order 3.

For the group with the ground field of characteristic $p>0$, we shall treat in the forthcoming paper.

1. Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field K of characteristic 0, and let $A(\mathfrak{g})$ be the group of all automorphisms of $\mathfrak{g}$. Then $A(\mathfrak{g})$ is a linear algebraic group defined over K. We shall denote by G the irreducible component of the identity of $A(\mathfrak{g})$ and call it the adjoint group of $\mathfrak{g}$. It coincides with a simple group defined by C. Chevalley and its Lie algebra is the derivation algebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{g}$. We shall denote by $A(G)$ the group of all automorphisms (birational and biregular) of G, and by $I(G)$ the normal subgroup of $A(G)$ of all inner automorphisms of G.

We denote by P_r the additive group generated by the roots of $\mathfrak{g}$, by T the group of automorphisms of P_r, and by W the subgroup of T generated by the reflexions by roots. W is the Weyl-group which is a normal subgroup of T. We denote by S the subgroup of T of those elements, by which a fundamental root system transforms into itself. Then T is a semi-direct product of S and W and we can show that $A(\mathfrak{g})/G\simeq S$. If $\mathfrak{g}$ is of types (A_l) $l\geq 2$, (D_l) $l\geq 5$ and (E_6), S is the cyclic group of order 2, and if of type (D_4), S is the symmetric group of degree 3 and for other types of $\mathfrak{g}$, S has a unit element only.

Let $\sigma\in A(G)$, then the differential map $d\sigma$ of σ is an automorphism of $\mathfrak{g}$ and $\sigma\rightarrow d\sigma$ is a homomorphism of $A(G)$ into $A(\mathfrak{g})$, moreover this mapping is one-to-one and maps $I(G)$ onto G. So we can identify the group $A(G)/I(G)$ with a subgroup of $A(\mathfrak{g})/G$. Therefore we have $A(G)=I(G)$ except when $\mathfrak{g}$ is of types (A_l) $l\geq 2$, (D_l) $l\geq 4$, (E_6).

In the following we shall consider the case where $\mathfrak{g}$ is of types (A_l) $l\geq 2$, (D_l) $l\geq 4$, and (E_6).

2. In this section we shall determine the conjugate classes of involutions in the adjoint groups. Let $\mathfrak{H}$ be a Cartan-subgroup of G, and $\mathfrak{h}$ its Lie algebra which is identified with a Cartan-subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{h}+\sum_r \mathfrak{g}_r$ be the Cartan decomposition of $\mathfrak{g}$ and denote (X, Y) the Killing form on $\mathfrak{g}$. For any root r, there is H_r' such that $(H_r', H)=r(H)$ for all $H\in\mathfrak{h}$. We set $H_r=2/r(H_r')\cdot H_r'$. Let X_r be a base for $\mathfrak{g}_r$ such that $[X_r, X_{-r}]=H_r$. Then we have

Lemma 1. *$\sigma\in A(\mathfrak{g})$ is in $\mathfrak{H}$ if and only if $\sigma(H)=H$ for all $H\in\mathfrak{h}$. Moreover $\mathfrak{H}\simeq\mathrm{Hom}\,(P_r, K^*)$ where $\mathrm{Hom}\,(P_r, K^*)$ is the additive group of all homomorphisms of P_r into multiplicative group of K. If $\chi\in\mathrm{Hom}\,(P_r, K^*)$, we denote by $h(\chi)\in A(\mathfrak{g})$ the automorphism such that $X_r\to\chi(r)X_r$, and $H\to H$ for all $H\in\mathfrak{h}$.*

We shall fix a fundamental root system $a_1,\cdots, a_l$ of $\mathfrak{g}$. And when we set $a_i(H_{a_j})=-a_{ij}$, we consider these roots are numbered as follows:

1) If $\mathfrak{g}$ is of type (A_l) $l\geq 2$

$a_{ii+1}=a_{i+1i}=1$, $a_{ii}=-2$ for all i and $a_{ij}=0$ for other i, j.

2) If $\mathfrak{g}$ is of type (D_l) $l\geq 4$

$a_{ii+1}=a_{i+1i}=1$ for $i=1, 2,\cdots, l-2$, and $a_{ii}=2$ for all i

$a_{ij}=0$ for other i, j.

3) If $\mathfrak{g}$ is of type (E_6)

$a_{ii+1}=a_{i+1i}=1$ for $i=1, 2,\cdots, 5$ and $a_{36}=a_{63}=1$

$a_{ii}=-2$ for all i and $a_{ij}=0$ for other i, j.

We call an automorphism σ of $\mathfrak{g}$ involution if σ^2 is the identity automorphism. Then in $\mathfrak{H}$ there are 2^l involutions which are denoted by $h\ (i_1\cdots i_s)$, $1\leq i_1<\cdots<i_s\leq l$. Here, $h\ (i_1\cdots i_s)$ corresponds to $\chi\in\mathrm{Hom}\,(P_r, K^*)$ such that $\chi(a_{i_1})=\cdots=\chi(a_{i_s})=-1$ and $\chi(a_k)=1$ for all k not equal to $i_1,\cdots, i_s$. As for these involutions we have

Lemma 2. *The involutions of G are conjugate to one of the following:*

1) *If $\mathfrak{g}$ is of type (A_l), $l\geq 2$*

$h(1), h(2),\cdots, h(k)\quad k=[l+1/2]$.

2) *If $\mathfrak{g}$ is of type (D_l), $l\geq 5$, l is odd*

$h(1), h(2),\cdots, h(k), h(l)\quad k=[l/2]$.

3) *If $\mathfrak{g}$ is of type (D_l), $l\geq 4$, l is even*

$h(1), h(2), h(3),\cdots, h(k), h(l-1), h(l)\quad k=[l/2]$.

4) *If $\mathfrak{g}$ is of type (E_6)* $\quad h(1), h(2)$.

We denote by $\mathfrak{N}_i$ the normalizer of $h(i)$ in G, which is an algebraic subgroup of G. Let $(\mathfrak{N}_i)_0$ be the irreducible component of the identity of $\mathfrak{N}_i$ and $\mathfrak{n}_i$ be the Lie algebra of $(\mathfrak{N}_i)_0$. Then we have

Lemma 3. 1) *If* $\mathfrak{g}$ *is of type* (A_l), $l \geq 2$

$$\mathfrak{n}_i \simeq \mathfrak{a}_1 \oplus \mathfrak{g}(A_{i-1}) \oplus \mathfrak{g}(A_{l-i})$$

where $\mathfrak{a}_1$ *is a 1-dimensional abelian Lie algebra and* $\mathfrak{g}(A_i)$ *are simple Lie algebras of types* (A_i).

2) *If* $\mathfrak{g}$ *is of type* (D_l), $l \geq 4$

$$\mathfrak{n}_i \simeq \mathfrak{g}(D_i) \oplus \mathfrak{g}(D_{l-i}),\ i=1,2,\cdots,k,\ k=[l/2]$$

$$\mathfrak{n}_l \simeq \mathfrak{a}_1 \oplus \mathfrak{g}(A_{l-1})$$

where $\mathfrak{g}(D_1)=\mathfrak{a}_1$, $\mathfrak{g}(D_2)=\mathfrak{g}(A_1)\oplus\mathfrak{g}(A_1)$, $\mathfrak{g}(D_3)=\mathfrak{g}(A_3)$ *and* $\mathfrak{g}(D_i)$ *are simple Lie algebras of types* (D_i) *for* $i \geq 4$.

3) *If* $\mathfrak{g}$ *is of type* (E_6)

$$\mathfrak{n}_1 \simeq \mathfrak{a}_1 \oplus \mathfrak{g}(D_5),\quad \mathfrak{n}_2 \simeq \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_5).$$

If two involutions in G are conjugate each other, their normalizers are isomorphic. So we have from Lemmas 2, 3 that the involutions of Lemma 2 are non-conjugate each other and that they form a complete representative of non-conjugate involutions in G, except for the type (D_l), l is even. We may also see that this is true for this exceptional case.

3. Using the results of the last section, we shall prove the following

Theorem. *Let* $\mathfrak{g}$ *be a simple Lie algebra over an algebraically closed field of characteristic* 0 *and* G *be the adjoint group of* $\mathfrak{g}$. *We shall denote by* $A(G)$ *the group of all automorphisms (birational and biregular) of* G *and by* $I(G)$ *the normal subgroup of* $A(G)$ *of all inner automorphisms of* G. *Then* $A(G)=I(G)$ *if* $\mathfrak{g}$ *is not of type* (D_4). *And* $A(G)/I(G)$ *is the cyclic group of degree* 3 *if* $\mathfrak{g}$ *is of type* (D_4).

We have seen that the theorem holds for the type (A_1), so we prove the theorem by induction with respect to the rank of G, when $\mathfrak{g}$ is of type (A_l). Suppose that we have proved the theorem for the groups with the rank less than l. Let $\sigma \in A(G)$ and we shall show that σ is the identity automorphism except for an inner automorphism. $\mathfrak{H}$ and $\mathfrak{H}^\sigma$ are conjugate by an inner automorphism. Therefore we may suppose that $\mathfrak{H}^\sigma=\mathfrak{H}$. Now $h(1)^\sigma$ and $h(1)$ are conjugate by Lemmas 2 and 3, so we may also suppose that $h(1)^\sigma=h(1)$, then $\mathfrak{N}_1^\sigma=\mathfrak{N}_1$ and $(\mathfrak{N}_1)_0^\sigma=(\mathfrak{N}_1)_0$. Since the irreducible subgroup G_1 of $(\mathfrak{N}_1)_0$, whose Lie algebra is $\mathfrak{g}(A_{l-1})$ transforms into itself by σ, we may consider σ fixes the elements of G_1 by the induction hypotheses. We can also see easily that σ fixes the elements of irreducible subgroup A whose Lie algebra is $\mathfrak{a}_1$ and σ fixes at the same time the elements $x_1(t)=\exp t\,(ad\,X_{a_1})$ of G, adjoining the inner automorphism if necessary. Since G is generated by the elements $x_1(t)$ and $(\mathfrak{N}_1)_0$, the theorem is proved. For the groups of types (D_l) $l \geq 5$, (E_6), we can prove the theorem by the same way. For the group of type (D_4), the involutions $h(1)$, $h(3)$

and $h(4)$ are non-conjugate each other and their normalizers are isomorphic to $\mathfrak{a}_1 \oplus \mathfrak{g}(A_3)$. If the $\sigma^* \in A(\mathfrak{g})$ such that $\sigma^*(H_{a_1})=H_{a_1}$, $\sigma^*(H_{a_3})=H_{a_3}$ and $\sigma^*(H_{a_4})=H_{a_4}$ is the differential of an automorphism σ of G, then $h(1)^\sigma=h(1)$ and $(\mathfrak{N}_1)_0^\sigma=(\mathfrak{N}_1)_0$. Since the automorphism of the group of type (A_3) is all inner, σ might be inner by the same consequence of the above proof. This contradicts that $d\sigma$ is not inner, and we have that $A(G)/I(G)$ is a subgroup of order 3. On the other hand, $G \simeq P\Omega_8(K, f)$, where f is a quadratic form of maximal index (cf. [1] or [4]), and Dieudonné has noted that there is an outer automorphism of order 3 of $P\Omega_8(K, f)$ (cf. [3, p. 64]). So we have $A(G)/I(G)$ is the cyclic group of order 3.

References

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