55. *On the Singular Integrals.* II

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1. The result about singular integrals for the non-periodic case, properly modified, can be extended to the periodic functions, as was proved by A. P. Calderón-A. Zygmund.

Let $x=(\xi_1,\cdots,\xi_n)$, $y=(\eta_1,\cdots,\eta_n),\cdots$ denote points in the n-dimensional Euclidean space E^n. By x we shall also denote the vector joining the origin $O=(0,\cdots,0)$ with the points x. The length of the vector x will be denoted by $|x|$. If $x\neq 0$, by x' we shall mean the projection of x onto the unit sphere Σ having O for center. Thus $x'=x/|x|$, $|x'|=1$.

Let us now consider a system $e_1, e_2,\cdots, e_n$ of independent vectors in E^n and let $x_0=O, x_1,\cdots$ be the sequence of all lattice points generated by these vectors

(1.1) $$x_i=m_1e_1+m_2e_2+\cdots+m_ne_n$$

where $m_1, m_2,\cdots, m_n$ are arbitrary integers. If we define $K^*(x)$ by the formula

(1.2) $$K^*(x)=K(x)+\sum_{i=1}^{\infty}\{K(x+x_i)-K(x_i)\},$$

where the kernel $K(x)$ satisfies the following conditions:

(1.3) $$K(x)=\Omega(x')/|x|^n$$

and

(1.4) 1° *The integral of $\Omega(x')$ extended over Σ is zero,*

2° *$\Omega(x')$ is continuous on Σ and its modulus of continuity $\omega(\delta)$ satisfies the Dini condition*

$$\int_0^1\omega(\delta)\frac{d\delta}{\delta}=\int_1^\infty\frac{1}{\delta}\omega\Big(\frac{1}{\delta}\Big)d\delta<\infty.$$

The series on the right of (1.2) converges absolutely and uniformly over any bounded set provided that we discard the first few terms.

For if $x\in S$ of any bounded set and i is large enough, then

(1.5) $$|K(x+x_i)-K(x_i)|\leqq\frac{A}{|x_i|^n}\omega\Big(\frac{c}{|x_i|}\Big)$$

and condition 2° imposed on the kernel K implies that the terms on the right here form a convergent series. The property (1.5) is used repeatedly (see [2, p. 252]).

Let us now suppose for the sake of simplicity that the vectors $e_1, e_2,\cdots, e_n$ are all mutually orthogonal and of length 1. We may assume that they are situated on coordinate axes. Let $f(x)=f(\xi_1,\cdots,$

$\xi_i, \cdots, \xi_n$) be a function of period 1 in each ξ_i and integrable over every bounded set. We shall consider the convolution

$$\tilde{f}^*(x)=\int_R f(y)K^*(x-y)\,dy=\int_R K^*(y)f(x-y)\,dy, \tag{1.6}$$

where R is the fundamental cube:

$$|\xi_i|\leqq\frac{1}{2}\quad(i=1,2,\cdots,n) \tag{R}$$

and the integral is taken in the principal value sense.

Some of the results they proved are the following:

Theorem 1. *If $f\in L$, then the measure of the set E_y of points $x\in R$ at which $|\tilde{f}^*|\geqq y>0$, satisfies the inequality*

$$|E_y|\leqq\frac{A}{y}\int_R |f|\,dx. \tag{1.7}$$

Theorem 2. *If $f\in L^p$, $1<p<\infty$, then $\tilde{f}^*$ also belongs to L^p and*

$$\int_R |\tilde{f}^*|^p\,dx\leqq A_p\int_R |f|^p\,dx. \tag{1.8}$$

Theorem 3. *If $|f|\log^+|f|$ is integrable over R, then*

$$\int_R |\tilde{f}^*|\,dx\leqq A\int_R |f|\log^+|f|\,dx+B. \tag{1.9}$$

Theorem 4. *According as $f\in L^p$, $1<p<\infty$ and $|f|\log^+|f|\in L$, the $\tilde{f}^*$ in* (1.8) *and* (1.9) *can be replaced by*

$$\tilde{F}^*(x)=\sup_{0<\varepsilon\leqq\frac{1}{2}}\left|\int_R f(x-y)\,K_\varepsilon^*(y)\,dy\right|, \tag{1.10}$$

where K_ε^* is the function equal to K^* except in the ε-neighbourhoods of the lattice points x_i, in which it is equal to zero.

Theorem 1 plays an essential role in our theory of singular integrals from a point of view of the theory of the interpolation of operation and can be described in the more extensive form. That is:

Theorem 5. *Let $f(x)$ belong to L, then the operation*

$$\tilde{f}_\varepsilon(x)=\int_{E^n} K_\varepsilon(x-y)f(y)\,dy \tag{1.11}$$

is of weak type (1, 1), *further the positive number ε can be replaced by any positive measurable function of x.*

This is proved by Lemmas 1 and 2 of Chap. I and Lemma 3 of Chap. II of their paper [1]. This enables us to treat easily the existence of the limit of (1.11) and the mean convergence of $\tilde{f}_\varepsilon$ to $\tilde{f}$ in each functional space.

It can be extended naturally to the more extensive class of functions. Let $\varphi(u)$ be a positive continuous increasing function for $u\geqq 0$ and satisfy the following properties: $\varphi(0)=0$,

$$\varphi(2u)=O(\varphi(u)), \tag{1.12}$$

$$\int_u^\infty \frac{\varphi(t)}{t^{r+1}}\,dt = O\left(\frac{\varphi(u)}{u^r}\right) \quad (1<r<\infty), \tag{1.13}$$

$$\int_1^u \frac{\varphi(t)}{t^2}\,dt = O\left(\frac{\varphi(u)}{u}\right), \tag{1.14}$$

for $u\to\infty$, and let $\chi(u)$ be equal to 0 in a right-hand neighbourhood of $u=0$, say for $u\leqq 1$, positive increasing function elsewhere and satisfy the following properties:

$$\chi(2u)=O(\chi(u)), \tag{1.15}$$

$$\int_u^\infty \frac{\chi(t)}{t^{r+1}}\,dt = O\left(\frac{\chi(u)}{u^r}\right) \quad (1<r<\infty), \tag{1.16}$$

and let us define the $X^*(u)$ by the equation:

$$\chi^*(u)=u\int_1^u t^{-2}\chi(t)\,dt. \tag{1.17}$$

Let L^φ and L^{χ^*} be the class of functions such that $f(x)$ in R is measurable and $\varphi(|f|)$ and $\chi^*(|f|)$ are integrable in R respectively. These classes contain the L^p $(1<p<r)$ and the Zygmund class of Theorem 3 respectively.

Now we have the following

Theorem 6. *If $f\in L$, then the operation $Tf=\tilde{f}_\varepsilon^*$ is of weak type $(1,1)$, that is, the measure of the set $E_y[|\tilde{f}_\varepsilon^*|]$ of points $x\in R$ at which $|\tilde{f}_\varepsilon^*|\geqq y>0$, satisfies the inequality*

$$|E_y[|\tilde{f}_\varepsilon^*|]\leqq \frac{A}{y}\int_R |f|\,dx, \tag{1.18}$$

where the constant A is independent of f and ε.

Further the positive number ε can be replaced by any non-negative measurable function of x such as $0<\varepsilon(x)\leqq \frac{1}{2}$.

Theorem 7. *If $f\in L^\varphi$, then f^* also belongs to L^φ and*

$$\int_R \varphi(|\tilde{f}^*|)\,dx \leqq A\int_R \varphi(|f|)\,dx+B. \tag{1.19}$$

Theorem 8. *If $f\in L^{\chi^*}$, then f^* belongs to L_χ and*

$$\int_R \chi(|\tilde{f}^*|)\,dx \leqq A\int_R \chi^*(|f|)\,dx+B. \tag{1.20}$$

Theorem 9. *According as $f\in L^\varphi$ and $f\in L^{\chi^*}$, the $\tilde{f}^*$ in* (1.19) *and* (1.20) *can be replaced by the $\tilde{F}^*$ respectively.*

2. For the proof of these theorems we quote the theory of the interpolation of the quasi-linear operation due to A. Zygmund [4]:

Theorem 10. *Suppose that $\mu(R)$ and $\nu(S)$ are both finite, and that the quasi-linear operation $h=Tf$ is of weak types (a,a) and (b,b), where $1\leqq a<b<\infty$. Suppose also that $\varphi(u)$, $u\geqq 0$, is a continuous increasing function satisfying the condition $\varphi(0)=0$ and*

$$\varphi(2u)=O(\varphi(u)), \tag{2.1}$$

$$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}\,dt=O\left(\frac{\varphi(u)}{u^b}\right), \tag{2.2}$$

$$\int_1^u \frac{\varphi(t)}{t^{a+1}}\,dt=O\left(\frac{\varphi(u)}{u^a}\right), \tag{2.3}$$

for $u\to\infty$. Then $h=Tf$ is defined for every f, where $\varphi(|f|)$ is μ-integrable and

$$\int_S \varphi(|h|)\,d\nu \leqq A\int_R \varphi(|f|)\,d\mu+B. \tag{2.4}$$

Theorem 11. *Suppose that $\mu(R)$ and $\nu(S)$ are finite, that $1\leqq a<b<\infty$, and that $h=Tf$ is a quasi-linear operation simultaneously of weak types (a,a) and (b,b). Let $\chi(u)$, $u\geqq 0$ be equal to 0 in a right-hand neighbourhood of $u=0$, say for $u\leqq 1$, positive and increasing elsewhere, satisfying*

$$\chi(2u)=O(\chi(u)), \tag{2.5}$$

$$\int_u^\infty \frac{\chi(t)}{t^{b+1}}\,dt=O\left(\frac{\chi(u)}{u^b}\right). \tag{2.6}$$

Write

$$\chi^*(u)=u^a\int_1^u t^{-a-1}\chi(t)\,dt. \tag{2.7}$$

Then $h=Tf$ is defined for all f belong to L^{χ^*} and we have

$$\int_S \chi(|h|)\,d\nu \leqq A\int_R \chi^*(|f|)\,d\mu+B. \tag{2.8}$$

3. Proof of the theorem. First Theorem 6 is proved by the above theorem and the following lemma due to A. P. Calderón-A. Zygmund [2, p. 254].

Lemma 1. *Let R_1 denote the cube*

$$|\xi_i|\leqq 1 \quad (i=1,2,\cdots,n) \tag{R$_1$}$$

and let f_1 be the function equal to f in R_1 and to zero elsewhere, then there exists a constant A independent of f and ε such that

$$|\tilde{f}_\varepsilon^*-\tilde{f}_1|\leqq A\int_R |f|\,dx, \quad \text{for} \quad x\in R. \tag{3.1}$$

We observe that the positive constant ε can be replaced by any measurable function of x such as $0<\varepsilon(x)\leqq \frac{1}{2}$.

PROOF OF THEOREM 9. It is now followed immediately by Theorems 2,5 and 10; Theorems 2,5 and 11 with Lemma 1 respectively.

Theorems 7 and 8 are followed by Theorem 9.

4. In this section we prove few results about the discrete analogous of the Hilbert transform.

Let us again consider in E^n, and a kernel $K(x)=\Omega(x')/|x|^n$ with properties described in Section 1. Let $e_1, e_2, \cdots, e_n$ be a system of n linearly independent vectors in E^n, thus in particular, all the e_i are different from zero. Let $p_0=O, p_1, \cdots$ be the sequence of all lattice points in E^n generated by this system, that is

$$p_i=m_1e_1+m_2e_2+\cdots+m_ne_n \tag{4.1}$$

where the coefficients m_i are arbitrary real integers. For any sequence $X=(x_0, x_1, \cdots)$ of real or complex numbers we define the transform $\widetilde{X}=(\widetilde{x}_0, \widetilde{x}_1, \cdots)$ by the formula

$$\widetilde{x}_i=\sum_j{}' x_j K(p_i-p_j), \tag{4.2}$$

the prime indicating that the term $j=i$ is omitted in summation.

The class of sequences X with

$$\|X\|_\varphi=\varphi^{-1}\Big(\sum_{i=0}^{\infty}\varphi(|x_i|)\Big)<\infty, \tag{4.3}$$

will be denoted by l^φ. The $\varphi(u)$ is the function which was introduced in the preceding paper [3, Theorem 2], and φ^{-1} is the inverse of φ. If $\varphi=u^p$, $p>1$, then the class l^φ is identical with the class l^p.

Then A. P. Calderón-A. Zygmund [2] have also proved

Theorem 12. *If X is in l^p, $p>1$, so is $\widetilde{X}$, and*

$$\|\widetilde{X}\|_p \leqq A_p\|X\|_p \tag{4.4}$$

where A_p depends only on p and the kernel K.

We can extend this for the class l^φ:

Theorem 13. *If X is in l^φ, so is $\widetilde{X}$ and*

$$\|\widetilde{X}\|_\varphi \leqq A\|X\|_\varphi. \tag{4.5}$$

5. For the sake of simplicity, we assume also that $e_1, e_2, \cdots, e_n$ are all mutually orthogonal, of length 1, and situated on the coordinate axes. Let R_i denote the cube with center p_i and edges of length 1, parallel to the axes. By R_i' we shall denote the concentric and similarly situated cube with edges of length 1/2. Given a sequence $X=(x_0, x_1, \cdots)$; let f denote the function taking the value x_i at the points R_i' $(i=0, 1, \cdots)$ and equal to zero elsewhere in E^n. The function f is in L^φ if and only if it is on l^φ, and the ratio $\|X\|_\varphi/\|f\|_\varphi$ depends on n and φ only.

First we prove the following

Theorem 14. *The operation with which X corresponds to $\widetilde{X}$ is of quasi-linear and weak type* $(1,1)$, *provided that the measure $E_y[|h|]$—the set of points $x\in E^n$ such as $|Tf|>y$—is replaced by the number of $\widetilde{x}_i$ at which $|\widetilde{x}_i|>y$.*

PROOF OF THEOREM 14. In the arguments of the proof of Theorem 14 [2], they proved:

$$|\widetilde{x}_i|\leqq 2^n|\widetilde{f}(x)|+2^n|x_i|\,|\widetilde{c}_i(x)|+A2^n|r_i|, \tag{5.1}$$

where

$$r_i = \sum_{j=0}^{\infty}{}' |x_j| |p_i - p_j|^{-n} \omega(|p_i - p_j|^{-1}), \tag{5.2}$$

$f(x)$ denotes the function taking the value x_i at the points R'_i ($i=0, 1, 2, \cdots$) and equal to zero elsewhere in E^n and $c_i(x)$ is the characteristic function of the set R'_i.

Now we show that the three terms of right-hand side of (5.1) are of weak type (1, 1) respectively.
In the first we have by Theorem 1

$$|\{x \mid |\tilde{f}| > y\}| \leqq \frac{M}{y} \int_{E^n} |f| \, dx = \frac{M}{y} \sum_{i=0}^{\infty} |x_i|. \tag{5.3}$$

Secondly we have by the weak type (1, 1) of $\tilde{\varphi}_i(x)$

$$\sum_{i=0}^{\infty} |\{x \mid |x_i \tilde{\varphi}_i(x)| > y\}| \leqq M \sum_{i=0}^{\infty} \frac{|x_i|}{y} \int_{E^n} |\varphi_i| \, dx = \frac{M}{y} \sum_{i=0}^{\infty} |x_i|. \tag{5.4}$$

Finally, the numbers of r_i at which $|r_i| > y$ are less than

$$\frac{M}{y} \sum_{|r_i| > y} |r_i| \leqq \frac{M}{y} \sum_{i=0}^{\infty} |r_i| = \frac{M}{y} \sum_{i=0}^{\infty} \Big(\sum_{j=0}^{\infty} |x_j| \alpha_{i-j} \Big), \tag{5.5}$$

where $\alpha_0 = 0$, $\alpha_i = |p_i|^{-n} \omega(|p_i|^{-1})$ for $i > 0$ and

$$\sum_{i=0}^{\infty} \alpha_i \leqq M \int_1^{\infty} \frac{1}{r} \omega\Big(\frac{1}{r}\Big) dr < \infty \tag{5.6}$$

by the property 2° of (1.4).
Thus by the above formulae, the numbers $\tilde{x}_i$ at which $|\tilde{x}_i| > y$ for any real positive y are less than $y^{-1} \sum_{i=0}^{\infty} |x_i|$ with constant multiple. We have proved Theorem 14 completely.

PROOF OF THEOREM 13. It is proved by Theorem 14 and the theorem of interpolation of the operation in the preceding paper [3, Theorem 4].

References

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