46. On the Singular Integrals. I

By Sumiyuki Koizumi
Department of Mathematics, Hokkaidô University, Sapporo, Japan
(Comm. by Z. Suetuna, M.J.A., April 12, 1958)

1. **Introduction.** Let E^n be an n-dimensional Euclidean space. Let $f(P)$ be a function L^p in E^n, $p\geqq 1$, and the kernel K has the form

(1.1) $$K(P-Q)=|P-Q|^{-n}\Omega[(P-Q)|P-Q|^{-1}],$$

where $\Omega(P)$ is a function defined on $\sum$ and satisfies the following conditions:

(1.2) $$\int_{\Sigma}\Omega(P)d\sigma=0,$$

where $d\sigma$ is the area element on $\sum$ and $\sum$ denotes a surface of the sphere of radius 1 with center at the origin,

(1.3) $$|\Omega(P)-\Omega(Q)|\leqq\omega(|P-Q|),$$

and $\omega(t)$ is an increasing function such that $\omega(t)\geqq t$ and

(1.4) $$\int_0^1\omega(t)\frac{dt}{t}=\int_1^\infty\omega\left(\frac{1}{t}\right)\frac{dt}{t}<\infty.$$

Now we define the operation T by

(1.5) $$Tf=\tilde{f}_\lambda(P)=\int_{E^n}K_\lambda(P-Q)f(Q)dQ,$$

where

(1.6) $$K_\lambda(P-Q)=\begin{cases}K(P-Q) & \text{if } |P-Q|\geqq 1/\lambda,\\ 0 & \text{elsewhere,}\end{cases}$$

and dQ is the volume element of E^n.

Then A. P. Calderón and A. Zygmund [1] (cf. also [4]) have proved the following

Theorem 1. *Let $f(P)$ belong to L^p, $1\leqq p<\infty$. Then*

(1.7) $$\tilde{f}(P)=\lim_{\lambda\to\infty}\tilde{f}_\lambda(P)$$

exists a.e. If $1<p<\infty$, then we have also

(1.8) $$\|\tilde{f}_\lambda\|_p\leqq A_p\|f\|_p,\quad \|\tilde{f}\|_p\leqq A_p\|f\|_p,$$

(1.9) $$\lim_{\lambda\to\infty}\|\tilde{f}-\tilde{f}_\lambda\|_p=0,$$

and

(1.10) $$\|\tilde{f}_*\|_p\leqq A_p\|f\|_p,\quad \textit{where}\quad \tilde{f}_*(P)=\sup_{\lambda>0}\tilde{f}_\lambda(P).$$

The constant A_p depends on p and the kernel K only.

We can extend this theorem for the class L^φ such that $f(P)$ is measurable and $\varphi(|f|)$ is integrable. $\varphi(u)$ is a continuous increasing function for $u\geqq 0$ and satisfies the following conditions: $\varphi(0)=0$, and

(1.11) $$\varphi(2u)=O(\varphi(u)),$$

(1.12) $$\int_u^\infty \frac{\varphi(t)}{t^{r+1}}dt=O\left(\frac{\varphi(u)}{u^r}\right) \quad (1<r<\infty),$$

(1.13) $$\int_1^u \frac{\varphi(t)}{t^2}dt=O\left(\frac{\varphi(u)}{u}\right)$$

for $u\to\infty$, and

(1.14) $$\varphi(2u)=O(\varphi(u)),$$

(1.15) $$\int_u^1 \frac{\varphi(t)}{t^{r+1}}dt=O\left(\frac{\varphi(u)}{u^r}\right), \quad (1<r<\infty),$$

(1.16) $$\int_0^u \frac{\varphi(t)}{t^2}dt=O\left(\frac{\varphi(u)}{u}\right),$$

for $u\to 0$.

In particular, these conditions are satisfied if $\varphi(u)=u^p$ or $u^p\psi(u)$, $1<p<r$ and $\psi(u)$ is a slowly varying function both for $u\to 0$ and $u\to\infty$.

Then we have

Theorem 2. *Let $f(P)$ belong to L^φ with this $\varphi(u)$. Then*

(1.17) $$\tilde{f}(P)=\lim_{\lambda\to\infty}\tilde{f}_\lambda(P)$$

exists a.e. We have also

(1.18) $$\|\tilde{f}_\lambda\|_\varphi \leqq A_\varphi \|f\|_\varphi, \quad \|\tilde{f}\|_\varphi \leqq A_\varphi \|f\|_\varphi,$$

(1.19) $$\lim_{\lambda\to\infty}\|\tilde{f}-\tilde{f}_\lambda\|_\varphi=0,$$

and

(1.20) $$\|\tilde{f}_*\|_\varphi \leqq A_\varphi \|f\|_\varphi, \quad \textit{where} \quad \tilde{f}_*(P)=\sup_{\lambda>0}\tilde{f}_\lambda(P).$$

The constant A_φ depends on the φ and the kernel K only.

2. Interpolation of the operation. The proof of Theorem 2 depends on the interpolation of the quasi-linear operation due to J. Marcinkiewicz [2] and A. Zygmund [3]. Let R and S be two spaces —for simplicity Euclidean spaces—with non-negative and completely additive measures μ and ν respectively.

Then J. Marcinkiewicz and A. Zygmund have proved

Theorem 3. *Suppose that $\mu(R)$ and $\nu(S)$ are finite, that $1\leqq a<b<\infty$ and that $h=Tf$ is a quasi-linear operation simultaneously of weak types (a, a) and (b, b). Suppose also that $\varphi(u)$ is a continuous increasing function for $u\geqq 0$ and satisfies the following conditions: $\varphi(0)=0$ and*

(2.1) $$\varphi(2u)=O(\varphi(u)),$$

(2.2) $$\int_u^\infty \frac{\varphi(t)}{t^{b+1}}dt=O\left(\frac{\varphi(u)}{u^b}\right),$$

(2.3) $$\int_1^u \frac{\varphi(t)}{t^{a+1}}dt=O\left(\frac{\varphi(u)}{u^a}\right),$$

for $u\to\infty$. *Then* $h=Tf$ *is defined for every* f *such that* $\varphi(|f|)$ *is* μ*-integrable, and we have*

$$\int_S \varphi(|h|)d\nu \leqq A\int_R \varphi(|f|)d\mu+B, \tag{2.4}$$

where the A, B *are independent of* f.

We now extend this theorem in the case where $\mu(R)$ and $\nu(S)$ are both infinite:

Theorem 4. *Suppose that* $\mu(R)$ *and* $\nu(S)$ *are both infinite, and that a quasi-linear operation* $h=Tf$ *is of weak types* (a,a) *and* (b,b), *where* $1\leqq a<b<\infty$. *Suppose also that* $\varphi(u)$ *is a continuous increasing function for* $u\geqq 0$ *satisfying the conditions:* $\varphi(0)=0$, (2.1), (2.2), (2.3) *for* $u\to\infty$ *and further*

$$\varphi(2u)=O(\varphi(u)), \tag{2.5}$$

$$\int_u^1 \frac{\varphi(t)}{t^{b+1}}dt=O\left(\frac{\varphi(u)}{u^b}\right), \tag{2.6}$$

$$\int_0^u \frac{\varphi(t)}{t^{a+1}}dt=O\left(\frac{\varphi(u)}{u^a}\right), \tag{2.7}$$

for $u\to 0$. *Then* $h=Tf$ *is defined for every* f *such that* $\varphi(|f|)$ *is* μ*-integrable, and we have*

$$\int_S \varphi(|h|)d\nu \leqq A\int_R \varphi(|f|)d\mu, \tag{2.8}$$

where A *is independent of* f.

The existence of this theorem is indicated by A. Zygmund [3] implicitly.

Proof of Theorem 4. Let f be any μ-measurable function on R such that $\varphi(|f|)$ is μ-integrable, and let $n(y)$ be the distribution function of $|h|$, that is the ν-measure of the set $E_y[|h|]=\{x\,|\,|h(x)|>y,\ x\in S\}$. Then we have

$$\int_0^\infty n(y)d\varphi(y)\leqq \sum_{j=-\infty}^{\infty}\eta_j\{\varphi(\lambda 2^{j+1})-\varphi(\lambda 2^j)\}=\sum_{j=-\infty}^{\infty}\eta_j\delta_j,$$

where $\eta_j=n(\lambda 2^j)$, $\delta_j=\varphi(\lambda 2^{j+1})-\varphi(\lambda 2^j)$ and $\lambda=3\kappa^2$.

For each fixed positive j we write

$$f=f_1+f_2+f_3,$$

where

$$\begin{aligned} f_1&=f & 1&\leqq|f|<2^j, & &=0 & &\text{elsewhere},\\ f_2&=f & 2^j&\leqq|f|, & &=0 & &\text{elsewhere},\\ f_3&=f & 0&\leqq|f|<1, & &=0 & &\text{elsewhere},\end{aligned}$$

and

$$h=Tf,\qquad h_i=Tf_i,\ i=1,2,3.$$

Then we have $f_1\in L^a\cup L^b$, $f_2\in L^a$, $f_3\in L^b$ respectively, and Tf is defined. Since

$$E_{\lambda 2^j}[|h|]\subset\bigcup_{i=1}^{3}E_{2^j}[|h_i|],\qquad \lambda=3\kappa^2\ (\kappa\geqq 1),$$

we have

$$\eta_j < M\left\{2^{-jb}\int_R |f_1|^b d\mu + 2^{-ja}\int_R |f_2|^a d\mu + 2^{-jb}\int_R |f_3|^b d\mu\right\}$$

and

$$\sum_{j=0}^{\infty} \eta_j \delta_j < M(S_1+S_2+S_3), \quad \text{say.}$$

Let ε_i be the μ-measure of the set of such points x in R that $2^{i-1} \leqq f < 2^i$ $(i=0, \pm 1, \pm 2, \cdots)$, then we have from (2.1) and (2.2)

$$S_1 \leqq A\int_{R_2} \varphi(|f|)d\mu, \qquad R_2=\{x \mid |f(x)| \geqq 1,\ x\in R\}.$$

Similarly from (2.1) and (2.3) we have

$$S_2 \leqq A\int_{R_2} \varphi(|f|)d\mu.$$

For S_3, we have by (2.2) and (2.6)

$$S_3 \leqq \int_{R_1} |f|^b d\mu \int_1^{\infty} \frac{\varphi(t)}{t^{b+1}}dt < A\int_{R_1} \varphi(|f|)d\mu,\ R_1=\{x \mid |f(x)|<1,\ x\in R\}.$$

Next for negative j, we write

$$f=f_4+f_5+f_6,$$

where

$$\begin{array}{llll} f_4=f & 2^j \leqq |f|<1, & =0 & \text{elsewhere,} \\ f_5=f & 0 \leqq |f|<2^j, & =0 & \text{elsewhere,} \\ f_6=f & 1 \leqq |f|, & =0 & \text{elsewhere,} \end{array}$$

and

$$h=Tf, \quad h_i=Tf_i, \quad i=4, 5, 6.$$

Then we have $f_4 \in L^a \cup L^b$, $f_5 \in L^b$, $f_6 \in L^a$ respectively, and Tf is defined, and we have

$$\eta_j \leqq M\left\{2^{-ja}\int_R |f_4|^a d\mu + 2^{-jb}\int_R |f_5|^b d\mu + 2^{-ja}\int_R |f_6|^a d\mu\right\},$$

and

$$\sum_{j=-\infty}^{-1} \eta_j \delta_j < M(S_4+S_5+S_6), \quad \text{say.}$$

And we have

$$S_4 \leqq A\int_{R_1} \varphi(|f|)d\mu,$$

by (2.5) and (2.7),

$$S_5 \leqq A\int_{R_1} \varphi(|f|)d\mu,$$

by (2.5) and (2.6),

$$S_6 \leqq A\int_R |f_6|^a d\mu \int_0^1 \frac{\varphi(t)}{t^{a+1}}dt \leqq A\int_{R_2} \varphi(|f|)d\mu,$$

by (2.7) and (2.3). Thus Theorem 4 is established.

3. Proof of Theorem 2

Proof of (1.17). By (1.13) and (1.15) we have

$$u \leqq A\varphi(u) \quad u \geqq 1,$$
$$u^p \leqq A_1\varphi(u) \quad 0 \leqq u < 1.$$

And we decompose f into the sum of the f_1 and f_2, where $f_1 = f$ if $|f| \geqq 1$, $=0$ elsewhere and $f_2 = f$ if $|f| \leqq 1$, $=0$ elsewhere. Then by (1.7), $\tilde{f}_1$ and $\tilde{f}_2$ exist a.e. and $\tilde{f}$ also does a.e.

Proof of (1.18). The first part now follows immediately by the application of Theorem 4 with (1.18) and the following lemma due to A. P. Calderón and A. Zygmund [1].

Lemma 1. *The operation* $Tf = \tilde{f}_\lambda$ *of* (1.5) *is of weak type* (1, 1) *or given an* $f \geqq 0$ *of* L^p, $p \geqq 1$ *and any number* $y > 0$, *there is a sequence of non-overlapping cubes* I_k *such that*

$$y \leqq \frac{1}{|I_k|} \int_{I_k} f(P)dP \leqq 2^n y, \quad (k=1, 2, \cdots),$$

and $f \leqq y$ *almost everywhere outside* $D_y = \bigcup_k I_k$. *Moreover* $|D_y| \leqq \beta^f(y)$ *and*

$$y \leqq \frac{1}{|D_y|} \int_{D_y} f(P)dP \leqq 2^n y.$$

And let $f \geqq 0$ *belong to* L^p, $1 \leqq p \leqq 2$, *in* E^n, *and let* E_y *be the set of points where the function* (1.5) *exceeds* y *in absolute value. Then*

$$|E_y| \leqq \frac{c_1}{y^2} \int_{E^n} [f(P)]_y^2 dP + c_2 \beta^f(y),$$

where $[f(P)]_y$ *denotes the function equal to* f *if* $f \leqq y$ *and equal to* y *otherwise, and* c_1 *and* c_2 *are constants independent of* λ.

The second part follows from the first part, (1.17) and the Fatou lemma.

Proof of (1.19). This is proved by the well-known process.

Proof of (1.20). The proof runs on the line of the arguments of Theorem 1 of Chap. II of A. P. Calderón and A. Zygmund [1]. We only indicate, using the same notation,

Lemma 2. *Let* $N(P)$ *be a function in* E^n *and suppose that*

$$|N(P)| \leqq \varphi(|P-O|)$$

where $\varphi(x)$ *is a decreasing function of* x *such that*

$$\int_{E^n} \varphi(|P-O|)dP < \infty.$$

Let $N_1(P)$ *be equal to* 1 *in the sphere of volume* 1 *and center at* O, *and zero elsewhere; and let* $\bar{f}(P)$ *be defined by*

$$\bar{f}(P) = \sup_\lambda \lambda^n \int_{E^n} N_1[\lambda(P-Q)]\,|f(Q)|\,dQ. \tag{3.1}$$

Then we have

$$\sup_{\lambda}\left|\lambda^n \int_{E^n} N[\lambda(P-Q)]f(Q)dQ\right| \leqq \bar{f}(P)\int_{E^n}\varphi(|P-O|)dP,$$

and the operation $Tf=\bar{f}$ *is of weak type* (1, 1) *and of strong type* (p, p), $(p>1)$.

Now we can apply Theorem 4 to (3.1), and lemmas which we need are obtained. We cease to go into further.

References

[1] A. P. Calderón and A. Zygmund: On the existence of certain integrals, Acta Math., **88**, 85-139 (1952).

[2] J. Marcinkiewicz: Sur l'interporation d'opération, C. R. Acad. Sci., Paris, **208**, 1217-1273 (1939).

[3] A. Zygmund: On a theorem of Marcinkiewicz concerning interpolation of operations, Jour. Math., **35**, 223-248 (1956).

[4] A. Zygmund: Hilbert transform in E^n, Proc. Int. Congress Math., Amsterdam, **3**, 140-151 (1954).