108. On Topological Properties of W^* algebras

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1. In this paper, we shall show some topological properties of W^*-algebras and their applications. Main assertions are as follows: (1) *Any closed invariant subspace of the adjoint space of C^*-algebras is algebraically spanned by positive functionals belonging to itself* [Theorem 1]. (2) *The direct product $M_1 \otimes M_2$ of W^*-algebras M_1 and M_2 is purely infinite, whenever either M_1 or M_2 is purely infinite* [Theorem 2]. This second assertion is the positive answer for a problem of J. Dixmier [4], and we can assert that all questions concerning the "type" of the direct product of W^*-algebras are now solved.

2. Let A be a C^*-algebra, $\tilde{A}$ the adjoint space of A. A subspace V of A is said invariant, if $f \in V$ means fa, $bf \in V$ for $a, b \in A$, where $\langle x, fa \rangle = \langle xa, f \rangle$ and $\langle x, bf \rangle = \langle bx, f \rangle$, where $\langle x, f \rangle$ is the value of f at $x(\in A)$.

Theorem 1.[1] *Any closed invariant subspace of $\tilde{A}$ is algebraically spanned by positive functionals belonging to itself.*

Proof. Let $\tilde{\tilde{A}}$ be the second adjoint space of A, then by Shermann's theorem (cf. [10]) $\tilde{\tilde{A}}$ is considered a W^*-algebra and A is a C^*-subalgebra of $\tilde{\tilde{A}}$, when it is canonically imbedded into $\tilde{\tilde{A}}$ as a Banach space.

Let V^0 be the polar of V in $\tilde{\tilde{A}}$, that is, $V^0 = \{a \mid |\langle a, f \rangle| \leqq 1, a \in \tilde{\tilde{A}}$ and all $f \in V\}$, then it is a $\sigma(\tilde{\tilde{A}}, \tilde{A})$-closed ideal of A, for $|\langle bac, V \rangle| = |\langle a, bVc \rangle| = |\langle a, V \rangle| \leqq 1$ for $a \in V^0$ and $b, c \in A$; hence $bac \in V^0$. Since A is $\sigma(\tilde{\tilde{A}}, \tilde{A})$-dense in $\tilde{\tilde{A}}$ and V^0 is $\sigma(\tilde{\tilde{A}}, \tilde{A})$-closed, this means $bac \in V^0$ for $b, c \in \tilde{\tilde{A}}$, so that V^0 is an ideal.

On the other hand, by a classical theorem of Banach spaces, the adjoint space of V is considered the quotient space $\tilde{\tilde{A}}/V^0$. Since $\tilde{\tilde{A}}/V^0$ is a C^*-algebra, by the author's theorem [8, 9] it is a W^*-algebra and $\sigma(\tilde{\tilde{A}}/V^0, V)$ is the σ-weak topology of $\tilde{\tilde{A}}/V^0$, that is, composed of all σ-weakly continuous linear functionals on $\tilde{\tilde{A}}/V^0$; hence by Dixmier's theorem [3] V is algebraically spanned by positive functionals belonging to itself. Moreover, since the positiveness of elements of V on $\tilde{\tilde{A}}/V^0$ means the positiveness on A, this completes the proof.

Now, let ν be a measure on a measure space and $L^1(\nu)$ and $L^\infty(\nu)$

be the spaces L^1 and L^∞ constructed on ν, then we know that $L^1(\nu)$ is $\sigma(L^1, L^\infty)$-sequentially complete. Applying Theorem 1, we will extend this to the following

Proposition 1.[2] *Let M be a W^*-algebra, M_* the Banach space of all σ-weakly continuous linear functionals on M, then M_* is $\sigma(M_*, M)$-sequentially complete.*

Proof. Let f be an element of the adjoint space $\widetilde{M}$ such that $\lim_n \langle x, f_n\rangle = \langle x, f\rangle$ for a sequence $(f_n) \subset M_*$ and all $x \in M$. Then particularly, $\lim_n \langle x, f_n\rangle = \langle x, f\rangle$ for $x \in uC$, where u is any unitary element and C is any maximal abelian self-adjoint subalgebra of M. Since uC is σ-weakly closed in M, it is considered the adjoint space of the restriction $(M_*)_{uC}$ of M_* on uC; hence by the author's theorem [8] $(M_*)_{uC}$ is an L^1-space as a Banach space. Therefore, since $(M_*)_{uC}$ is $\sigma((M_*)_{uC}, uC)$-sequentially complete by a classical theorem, the restriction $(f)_{uC}$ of f on uC belongs to $(M_*)_{uC}$; hence f is σ-weakly continuous on uC and analogously it is continuous on Cu. Now, let V be a subspace of all bounded linear functionals which are σ-weakly continuous on uC and Cu for all u and C, then it is closed in $\widetilde{M}$. Moreover, since every element of M is expressed as a finite linear combination of unitary elements, V is invariant; hence by Theorem 1 it is algebraically spanned by positive functionals belonging to itself. Suppose that a positive $\varphi \in V$, then it is σ-weakly continuous on every maximal abelian self-adjoint subalgebra, so that it is completely additive; hence by Dixmier's theorem [3] it is σ-weakly continuous on M, and so every element of V is also so. This completes the proof of Propoisition 1.

The above proposition has been proved by H. Umegaki [11] for a W^*-algebra of finite type.

Remark. Theorem 1 has some other applications; for example, it is of use in case which deals with "not necessarily adjoint preserving" homomorphisms of C^*-algebras.

Next we shall show an example of topological property which is negative in non-abelian case. Let I be a discrete locally compact space, and consider on I the measure which, to each point of I, attaches the mass $+1$. The corresponding L^p-spaces are denoted by l^p and the Banach space of complex valued continuous functions which vanish at

1) 2) Added in Proof. Combining Theorem 1 with a recent publication of A. Grothendieck [12], it implies that a Jordan decomposition is possible in any invariant closed subspace. Proposition 1 is more directly obtained from the result of Grothendieck — in fact, his result implies as a corollary that a bounded linear functional on a W^*-algebra is σ-weakly continuous if it is so on any maximal abelian self-adjoint subalgebra.

infinity by c_0. Then $\widetilde{c}_0=l^1$ and $\widetilde{l}^1=l^\infty$. Moreover, $\sigma(l^1, l^\infty)$-sequentially convergence in l^1 is equivalent to norm-convergence in l^1 [1, p. 137]. Now this fact has the following analogy [2, 7]: Let H be a Hilbert space, C the Banach space of all completely continuous linear operators on H, T the Banach space with the trace-norm of all trace-class linear operators on H and B the Banach space of all bounded linear operators, then $\widetilde{C}=T$ and $\widetilde{T}=B$. J. Dixmier [2] raised a question as follows: *Is $\sigma(T, B)$-sequential convergence in T equivalent to norm-convergence in it?*

We will show that the answer[3] for this is negative. Let H be a separable Hilbert space, (ψ_i) a complete orthonormal basis of H and e be the projection of H onto one-dimensional subspace $(\lambda\psi_1)$, then a vector space Be is a closed subspace of T. We consider matrix representation of Be. Then,

$$xe=\begin{Vmatrix} a_{11}\,0\,0\,0\,0\cdots \\ a_{21}\,0\,0\,0\,0\cdots \\ a_{31}\,0\,0\,0\,0\cdots \\ \vdots\ \cdots\cdots\cdots \\ \vdots\ \cdots\cdots\cdots \end{Vmatrix},\quad \mathrm{Tr}\,((ex^*xe)^{\frac{1}{2}})=\mathrm{Tr}\,\left(\begin{Vmatrix} \sum_{i=1}^{\infty}|a_{i1}|^2\,0\,0\,0\cdots \\ 0\quad 0\,0\,0\cdots \\ 0\quad 0\,0\,0\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{Vmatrix}^{\frac{1}{2}}\right)$$

$$=\mathrm{Tr}\left(\begin{Vmatrix} (\sum_{i=1}^{\infty}|a_{i1}|^2)^{\frac{1}{2}}\,0\,0\,0\cdots \\ 0\quad 0\,0\,0\cdots \\ 0\quad 0\,0\,0\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{Vmatrix}\right)=(\sum_{i=1}^{\infty}|a_{i1}|^2)^{\frac{1}{2}}.$$

Hence a closed subspace Be in T is isometric to a Hilbert space l^2. Let (x_ne) be a complete orthonormal basis in the Hilbert space Be, then $\mathrm{Tr}\,(yx_ne)=\mathrm{Tr}\,(eyx_ne)\to 0$ $(n\to\infty)$ for any $y\in B$, so that (x_ne) is $\sigma(T, B)$-convergent to 0. On the other hand, $\|x_ne\|_1=1$, where $\|.\|_1$ is the norm of T. Moreover, put $a_n=x_ne+ex_n^*$ and $b_n=ix_ne-iex_n^*$, then (a_n) and (b_n) are $\rho(T, B)$-convergent to 0. Suppose that $\|a_n\|_1\to 0$ and $\|b_n\|_1\to 0$, then $\|a_n-ib_n\|_1=2\|x_ne\|_1\to 0$, and this is a contradiction. Therefore, in T and more strongly in the self-adjoint portion of T, $\sigma(T, B)$-sequential convergence is not equivalent to norm-convergence.

3. Finally we shall notice a topological property which has a useful application. Let M be a semi-finite W^*-algebra in the sense of Dixmier [4], then by Dixmier's result there is a faithful normal semi-trace φ such that the algebraic span $\mathfrak{M}$ of $\{a\,|\,\varphi(a)<\infty,\ a\in M^+\}$ is a σ-weakly dense ideal in M, where M^+ is the positive portion of M.

Proposition 2. *Let e be a projection belonging to $\mathfrak{M}$, then the adjoint operation is strongly continuous on bounded spheres of Me.*

3) Added in Proof. J. Dixmier communicates to me that this example has been also constructed by A. Grothendieck.

Proof. Put $a\varphi(x)=\varphi(ax)$ $(a\in\mathfrak{M})$, then $\{a\varphi \mid a\in\mathfrak{M}\}$ is a total set of σ-weakly continuous linear functionals, that is, $a\varphi(x)=0$ for all $a\in\mathfrak{M}$ implies $x=0$. Suppose that $(x_\alpha e)$ $(\|x_\alpha e\|\leqq 1)$ converges strongly to 0, then $|a\varphi((x_\alpha e)(x_\alpha e)^*)|=|\varphi(ax_\alpha eex_\alpha^*)|=|\varphi(ex_\alpha^*ax_\alpha e)|=|e\varphi(ex_\alpha^*ax_\alpha e)|\leqq |e\varphi(ex_\alpha^*x_\alpha e)|^{\frac{1}{2}}|e\varphi(ex_\alpha^*a^*ax_\alpha e)|^{\frac{1}{2}}\to 0$; hence a bounded set $\{(x_\alpha e)(x_\alpha e)^*\}$ converges σ-weakly to 0, so that $\{(x_\alpha e)^*\}$ converges strongly to 0, which completes the proof.

The restriction "*Me*" in the above proposition is essential—in fact the adjoint operation is not strongly continuous on bounded spheres of eM as follows: let (ψ_i) be a complete orthonormal basis of an infinite dimensional Hilbert space H, (a_i) be a family of bounded operators such that $a_i\psi_i=\psi_1$ and $a_i\psi_j=0$ $(i\neq j)$, then $\|a_i\|=1$ and (a_i) is strongly convergent to 0. On the other hand, since $a_i^*\psi_1=\psi_i$, (a_i^*) is not strongly convergent to 0.

Proposition 2′. *Let M be a purely infinite W^*-algebra in the sense of Dixmier and let e be a non-zero projection of M, then the adjoint operation is not strongly continuous on bounded spheres of eMe.*

Proof. Since eMe is also purely infinite, it is enough to suppose that $e=I$, where I is the unit of M. Then since M contains a σ-weakly closed subalgebra N which is a factor of type I_∞, by the above consideration the adjoint operation is not strongly continuous on bounded spheres of M, which completes the proof.

Now we shall show an application of Propositions 2 and 2′ to the study of direct products of general W^*-algebras [4, 5]. J. Dixmier raises a problem concerning the direct product of W^*-algebras as follows: *Let M_1 and M_2 be W^*-algebras, one of which is purely infinite. Then, can we conclude that the direct product $M_1\otimes M_2$ is also purely infinite?* We show that the answer for this problem is positive.

Let M_1 and M_2 be W^*-algebras on Hilbert spaces H_1 and H_2 respectively. Then the direct product $M_1\otimes M_2$ is defined as the weak closure of the algebraic direct product on $H_1\otimes H_2$. For our purpose, we shall refer to the results and the notations of Murray-von Neumann [6, pp. 127–138]. Though these are obtained under the assumption of separability, it is easy to extend them to the general case.

Let B_1 and B_2 be the algebras of all bounded operators on Hilbert spaces H_1 and H_2, then the operator a in $H_1\otimes H_2$ can be represented by an operator matrix $\langle a_{\alpha,\beta}\rangle$ $(a_{\alpha,\beta}\in B_2)$ [6, Lemma 2.4.3]. Let M_2 be a W^*-algebra on H_2, then it is easily seen that the element b in $B_1\otimes M_2$ is expressed by $\langle b_{\alpha,\beta}\rangle$ $(b_{\alpha,\beta}\in M_2)$ under the above representation. Now put $P_\gamma(\langle b_{\alpha,\beta}\rangle)=\langle\delta_{\alpha,\beta}b_{\gamma,\gamma}\rangle$ for all γ, where $\delta_{\alpha,\beta}$ is the Kronecker's symbol, then P_γ are considered as linear mappings of $B_1\otimes M_2$ onto $I_1\otimes M_2$, where I_i $(i=1,2)$ is the unit of B_i, and we can show the following properties:

(1) $P_\gamma(I)=I$, where I is the unit of $B_1\otimes B_2$, (2) $\|P_\gamma(b)\|\leqq\|b\|$ (3) $P_\gamma(h)\geqq 0$ for $h\geqq 0$, (4) $P_\gamma(ubv)=uP(b)v$ for $u, v\in I_1\otimes M_2$, (5) $P_\gamma(b)^*P_\gamma(b)\leqq P_\gamma(b^*b)$, (6) P_γ are weakly and strongly continuous on bounded spheres, and (7) $P_\gamma(b^*b)=0$ for all γ imply $b=0$.

Since these properties are easily seen by a direct calculation according to the rules of computation in Murray-von Neumann's lemmas, we shall restrict the proof to (5)–(7).

(5) $P_\gamma(\langle b^*_{\beta,\alpha}\rangle\langle b_{\alpha,\beta}\rangle)=P_\gamma(\langle\sum_\zeta b^*_{\alpha,\zeta}b_{\beta,\zeta}\rangle)=\langle\delta_{\alpha,\beta}(\sum_\zeta b^*_{\gamma,\zeta}b_{\gamma,\zeta})\rangle\geqq\langle\delta_{\alpha,\beta}b^*_{\gamma,\gamma}b_{\gamma,\gamma}\rangle$ $=P_\gamma(b)^*P_\gamma(b)$.

(6) By the definition 2.4.2 of [6], $b\langle 0,0,\cdots,\psi_\gamma, 0,\cdots\rangle=\langle b_{\gamma,1}\psi_\gamma, b_{\gamma,2}\psi_\gamma,\cdots, b_{\gamma,\gamma}\psi_\gamma,\cdots\rangle$ and so the mappings $b\to b_{\gamma,\gamma}\to I_1\otimes b_{\gamma,\gamma}$ are weakly continuous on bounded spheres. Then, the weak continuity on bounded spheres and (5) imply the strong continuity on bounded spheres.

(7) Since $P_\gamma(b^*b)=\langle\delta_{\alpha,\beta}(\sum_\zeta b^*_{\gamma,\zeta}b_{\gamma,\zeta})\rangle$, $P_\gamma(b^*b)=0$ for all γ imply $b_{\gamma,\zeta}=0$ for all γ,ζ; hence $b=0$. This completes the proof.

Let M_1 be a W^*-algebra on H_1, then the direct product $M_1\otimes M_2$ is a subalgebra of $B_1\otimes M_2$, and so the restriction of P_γ to $M_1\otimes M_2$ defines a linear mapping of $M_1\otimes M_2$ onto $I_1\otimes M_2$. Therefore, we obtain

Proposition 3. *There is a family* (Q_γ) *of linear mappings of* $M_1\otimes M_2$ *onto* $I_1\otimes M_2$ *satisfying the properties* (1)–(7).

Theorem 2. *Let* M_1 *and* M_2 *be* W^**-algebras, one of which is purely infinite. Then the direct product* $M_1\otimes M_2$ *is also purely infinite.*

Proof. Suppose that M_2 is purely infinite and that there is a non-zero central projection z of $M_1\otimes M_2$ such that $(M_1\otimes M_2)z$ is semi-finite, and let $\mathfrak{M}$ be an ideal of $(M_1\otimes M_2)z$ in Proposition 2. By Proposition 3 there is a mapping Q_{γ_0} of $M_1\otimes M_2$ onto $I_1\otimes M_2$ such that $Q_{\gamma_0}(e)\neq 0$ for some projection $e\in\mathfrak{M}$. Since $Q_{\gamma_0}(e)>0$, there are a non-zero projection $p(\in I_1\otimes M_2)$ and a positive number $\lambda(>0)$ such that $\lambda p\leqq Q_{\gamma_0}(e)$.

Suppose that (x_α) ($\|x_\alpha\|\leqq 1$, $x_\alpha\in p(I_1\otimes M_2)p$) converges strongly to 0, then $(x_\alpha e)$ converges strongly to 0; hence by Proposition 2 (ex^*_α) converges strongly to 0, and so by the strong continuity on bounded spheres of Q_{γ_0}, $Q_{\gamma_0}(ex^*_\alpha)=Q_{\gamma_0}(e)x^*_\alpha$ converges strongly to 0, so that $(pQ_{\gamma_0}(e)p+I-p)^{-1}pQ_{\gamma_0}(e)x^*_\alpha=(pQ_{\gamma_0}(e)p+I-p)^{-1}pQ_{\gamma_0}(e)px^*_\alpha=x^*_\alpha$ converges strongly to 0. Therefore, the adjoint operation is strongly continuous on bounded spheres of $p(I_1\otimes M_2)p$, and this contradicts Proposition 2′, which completes the proof.

As J. Dixmier [4, l'exercise 4, p. 250] points out, Theorem 2 implies very easily the following

Corollary. *If* M_1 *is a continuous* W^**-algebra and* M_2 *an arbi-*

trary W^*-algebra, then the direct product $M_1 \otimes M_2$ is continuous.

Therefore, we can say that all questions concerning the "type" of the direct product of W^*-algebras are now solved.

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