127. On the B-covers in Lattices

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Let L be a lattice. For any two elements a and b of L we shall define the following three kinds of sets:

(1) $$J(a,b)=\{x \mid x=(a\frown x)\smile(b\frown x)\}$$

(2) $$C(J(a,b))=\{x \mid x=(a\smile x)\frown(b\smile x)\}$$

(3) $$B(a,b)=J(a,b)\frown C(J(a,b)).$$

$B(a,b)$ is called the B-cover of a and b. If $c\in B(a,b)$, we shall write acb simply.

In case L is a normed lattice, a point c is defined to be between two points a and b if $d(a,c)+d(c,b)=d(a,b)$, where $d(x,y)=|x\smile y|-|x\frown y|$. Several lattice characterizations of this metric betweeness have been obtained by V. Glivenko [1], L. M. Blumenthal and D. O. Ellis [2] and the author [3]; namely c lies between a and b in the metric sense if and only if one of the following conditions is satisfied in the associated normed lattice L.

(G) $$(a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$$

(G*) $$(a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$$

(G**) $$(a\smile c)\frown(b\smile c)=c=c\frown(a\smile b)$$

(M) $$(a\smile(b\frown c))\frown(b\smile c)=c.$$

Thus our definition of "acb" in an arbitrary lattice is a generalization of metric betweeness in a normed lattice. The notion of B-cover for a normed lattice is due to L. M. Kelley [4].

In Theorem 1 we shall assert that $(a]\smile(b]=J(a,b)\subset(a\smile b]$, $[a)\frown[b)=C(J(a,b))\subset[a\frown b)$. In Theorem 2 we shall deal with the relations between the two B-covers $B(a,b)$ and $B(b,c)$.

In Theorem 3 we shall consider the necessary and sufficient condition (A) in order that L be a distributive lattice.

In Theorems 4 and 5, we shall give the structures of $B(a,b)$ by imposing algebraic restrictions on them. Theorem 4 gives a generalization of the important result obtained by L. M. Kelley.

Now let $x\in J(a,b)$, then we have $x\geqq x\frown(a\smile b)\geqq(a\frown x)\smile(b\frown x)=x$, hence we obtain $x\frown(a\smile b)=(a\frown x)\smile(b\frown x)$, that is $(a,x,b)D$. From $x\frown(a\smile b)=x$, we get $x\leqq a\smile b$. We have clearly $a\smile b\in J(a,b)$, and $x\in J(a,b)$ if $x\leqq a$ or $x\leqq b$. On the other hand any element x of $J(a,b)$ is represented by $x=(a\frown x)\smile(b\frown x)$, where $a\frown x\in(a]$, $b\frown x\in(b]$. If we take any two elements x,y from $J(a,b)$, then $x\smile y$ belongs to $J(a,b)$. Indeed we have $x\smile y=(a\smile b)\frown(x\smile y)\geqq(a\frown(x\smile y))\smile(b\frown(x\smile y))$

$\geqq (a\frown x)\smile(a\frown y)\smile(b\frown x)\smile(b\frown y)=x\smile y$.

Similarly any element x of $C(J(a,b))$ is equal to or greater than $a\frown b$. Therefore we obtain

Theorem 1. In a lattice L we have

(1) $$(a]\smile(b]=J(a,b)\subset(a\smile b]$$

(2) $$[a)\frown[b)=C(J(a,b))\subset[a\frown b)$$

where $(x]=\{z\mid z\leqq x\}$, $A\smile B=\{x\smile y;\, x\in A, y\in B\}$ *if* $A, B\subset L$, *etc.*

Lemma 1. axb implies $x\frown(a\smile b)=x=x\smile(a\frown b)$.

Proof. By axb, we get $x\geqq x\frown(a\smile b)\geqq(x\frown a)\smile(x\frown b)=x$, $x\leqq x\smile(a\frown b)\leqq(x\smile a)\frown(x\smile b)=x$.

Lemma 1 shows that (G) implies (G*), (G**) in any lattice.

Lemma 2. axb implies $a\frown x\geqq a\frown b$, $a\smile b\geqq a\smile x$.

Proof. From Lemma 1, we have $a\frown b\leqq x\leqq a\smile b$. Therefore $a\frown x\geqq a\frown b$, $a\smile b\geqq a\smile x$.

Lemma 3. ax_ib $(i=1,2)$, ax_1x_2 imply x_1x_2b.

Proof. By ax_1x_2, ax_2b we have $x_2\geqq(x_1\frown x_2)\smile(x_2\frown b)\geqq(a\frown x_2)\smile(x_2\frown b)=x_2$. On the other hand, $x_2\leqq x_1\smile x_2\leqq a\smile x_2$ by Lemma 2, and hence we have $x_2\leqq(x_1\smile x_2)\frown(x_2\smile b)\leqq(a\smile x_2)\frown(x_2\smile b)=x_2$ by ax_2b.

Lemma 4. axb, byc, abc imply xby.

Proof. Since $a\frown b\leqq x\leqq a\smile b$, $b\frown c\leqq y\leqq b\smile c$ by axb, byc, we have $a\frown b\leqq b\frown x\leqq b$, $b\frown c\leqq b\frown y\leqq b$, and hence $(a\frown b)\smile(b\frown c)\leqq(b\frown x)\smile(b\frown y)\leqq b$. However $(a\frown b)\smile(b\frown c)=b$ by abc. Thus we obtain $(b\frown x)\smile(b\frown y)=b$. Similarly we have $b\leqq(b\smile x)\frown(b\smile y)\leqq(a\smile b)\frown(b\smile c)$ from $b\leqq b\smile x\leqq a\smile b$, $b\leqq b\smile y\leqq b\smile c$, and hence we have $(b\smile x)\frown(b\smile y)=b$ by abc.

Lemma 5. abc, axb, byc imply $a\frown y\leqq x\frown y$.

Proof. We have $x=x\smile(a\frown b)\geqq x\smile(a\frown y)\geqq x$ from axb, aby. Hence we have $x\smile(a\frown y)=x$ and then $a\frown y\leqq x\frown y$.

Lemma 6. (G) is equivalent to (G*) in a modular lattice. This proof was observed in L. M. Blumenthal [2].

Lemma 7. If L is modular, then $B(a,b)$ is a sublattice; in case L is not modular, $B(a,b)$ is not necessarily a sublattice.

Proof. If axb, ayb, then we have $x\smile y=(a\frown x)\smile(b\frown x)\smile(a\frown y)\smile(b\frown y)\leqq(a\frown(x\smile y))\smile(b\frown(x\smile y))\leqq(a\smile b)\frown(x\smile y)=x\smile y$. Since $x\smile(a\frown b)=x$, $y\smile(a\frown b)=y$, we have $(x\smile y)\smile(a\frown b)=x\smile y$, and hence $a(x\smile y)b$ by Lemma 6. Similarly we have $a(x\frown y)b$.

If L contains 8 elements $a, b, x, y, z, z_1, x_1, y_1$ such that $a\smile b>x_1>a$, $a\smile b>y_1>b$, $a>x>a\frown b$, $b>y>a\frown b$, $x_1\smile y_1=a\smile b$, $x_1\frown y_1=z_1>x\smile y=z$, $x\frown y=a\frown b$ (L is certainly non-modular in this case), then we have axb, ayb but not $a(x\smile y)b$.

Lemma 8. In case L is modular, abc, axb, byc imply axc, ayc.

Proof. By abc, axb we have $x=x\smile(a\frown b)\geqq x\smile(a\frown c)\geqq x$, and hence $x=x\smile(a\frown c)$. Since $a\frown b\leqq b\frown x$, $c\frown x\leqq b\frown x$ by axb, xbc, we have (1) $(a\frown b)\smile(c\frown x)\leqq b\frown x$. Since L is modular, we have $(x\frown c)\smile(a\frown b)$

$=x\frown(c\smile(a\frown b))$ from $a\frown b\leqq x$, and then $c\smile(a\frown b)\geqq(b\frown c)\smile(a\frown b)=b$ by abc, and hence we have (2) $(a\frown b)\smile(c\frown x)\geqq b\frown x$. From (1), (2) we have $(a\frown b)\smile(c\frown x)=b\frown x$. Thus we have $x=(a\frown x)\smile(b\frown x)=(a\frown x)\smile(a\frown b)\smile(c\frown x)=(a\frown x)\smile(c\frown x)$.

Accordingly we have $(a\frown x)\smile(c\frown x)=x=x\smile(a\frown c)$, that is axc by Lemma 6. We have ayc similarly.

Lemma 9. In case L is modular, abc, axb, byc imply xyc, axy.

Proof. From xbc, byc we have $y\leqq y\smile(x\frown c)\leqq y\smile(b\frown c)=y$. By ayc, Lemma 5, xby and byc, we have $y=(a\frown y)\smile(y\frown c)\leqq(x\frown y)\smile(y\frown c)\leqq(b\frown y)\smile(y\frown c)=y$. Hence we have $y=y\smile(x\frown c)=(x\frown y)\smile(y\frown c)$. Similarly we have axy.

Remark. In case L is non-modular, Lemmas 8 and 9 are not necessarily true. For, if L contains 6 elements $a, b, c, a\frown c, x, y$ such that $b>x>a>a\frown c$, $b>y>c>a\frown c$, $x\smile y=a\smile c=b$, $x\frown y=a\frown c$, when L is certainly non-modular, then we have abc, axb, byc but we have not axc, ayc, axy and xyc.

Theorem 2. If abc, axb, byc, then we have
(1) *xby in any lattice,*
(2) *axc, ayc, xyc, axy in a modular lattice.*

Corollary 2.1. In a modular lattice ax_ib $(i=1,2)$, byc, ax_1x_2 abc imply x_1x_2y.

Proof. Since ax_1y, ax_2y by Lemma 9 and we have x_1x_2b by Lemma 3, thus we have x_1by, x_2by by Lemma 4. We have further $x_2\leqq x_2\smile(x_1\frown y)\leqq x_2\smile(x_1\frown b)=x_2$ by x_1by, x_1x_2b, $x_2=(a\frown x_2)\smile(x_2\frown y)\leqq(x_1\frown x_2)\smile(x_2\frown y)$ by ax_2y, ax_1x_2, $\leqq(x_1\frown x_2)\smile(x_2\frown b)=x_2$ by x_2by, x_1x_2b. Hence we obtain x_1x_2y.

Corollary 2.2. In a modular lattice L, suppose that ax_ib $(i=1,2)$, byc, abc, ax_1x_2; then we have

$$(x_1\smile x_2)\frown y=(x_1\frown y)\smile(x_2\frown y).$$

Proof. We have x_1x_2y from Corollary 2.1, hence we have $(x_1\frown x_2)\smile(x_2\frown y)=x_2\frown(x_1\smile y)$. Since L is modular, we have the following equivalent equation $(x_1\smile x_2)\frown y=(x_1\frown y)\smile(x_2\frown y)$.

Theorem 3. In order that L be a distributive lattice it is necessary and sufficient that the condition (A) below hold for any elements a, b of L.

(A) $x\in B(a,b)$ if and only if $a\frown b\leqq x\leqq a\smile b$.

Proof. It is clear that if L is distributive then (A) holds for any elements a, b of L. Suppose that L is not distributive. Then there exist five elements a, b, c, d, e such that either

(α) $d=a\frown b=a\frown c=b\frown c,\ e=a\smile b=a\smile c=b\smile c$

or

(β) $d=a\frown b=a\frown c,\ e=a\smile b=a\smile c,\ d<b<c<e$.*)

*) Cf. G. Birkhoff: Lattice Theory, Theorem 2, 134 (1948).

In each case we have $d=a\frown b, e=a\smile b, d<c<e$. However we have $c\notin B(a,b)$; because in case (α) $(c\frown a)\smile(c\frown b)=d\smile d=d\neq c$, and in case ($\beta$) $(c\frown a)\smile(c\frown b)=a\smile d=a\neq c$. Thus if L is not distributive, the condition (A) dose not hold for some elements a, b of L. This proves Theorem 3.

In any lattice L

(1) axb, ayb, xay and xby imply $x\smile y=a\smile b, x\frown y=a\frown b$. For, we have $x\smile y\leqq a\smile b$ from $x\leqq a\smile b, y\leqq a\smile b$, and $a\smile b\leqq x\smile y$ similarly, and hence $a\smile b=x\smile y$. Similarlies $x\frown y=a\frown b$.

(2) $B(a,b)=B(c,d)$ implies $a\smile b=c\smile d, a\frown b=c\frown d$. For, it is evident from (1).

If L is distributive, then Theorem 3 shows that $a\smile b=c\smile d$, $a\frown b=c\frown d$ imply $B(a,b)=B(c,d)$.

Theorem 4. For any elements a, b, c, d of L

(1) $B(a,b)=B(c,d)$ implies $a\smile b=c\smile d, a\frown b=c\frown d$ in any lattice L.

(2) $a\smile b=c\smile d, a\frown b=c\frown d$ imply $B(a,b)=B(c,d)$, if and only if L is a distributive lattice.

Proof. It is sufficient to prove that if L is not distributive there exist four elements a, b, x, y such that $a\smile b=x\smile y$, $a\frown b=x\frown y$, but $B(a,b)\neq B(x,y)$. As is shown in the proof of Theorem 3 there exist five elements a, b, c, d, e such that either (α) or (β) holds. If we put $x=a$, $y=c$, then we have $d=a\frown b=x\frown y$, $e=a\smile b=x\smile y$, but $c\notin B(a,b)$, $c\in B(x,y)=B(a,c)$.

Corollary 4.1. In any lattice, suppose that $B(a,b)=B(x_1,x_2)$, ax_ib $(i=1,2)$, ax_1x_2. Then we have $x_1=a$, $x_2=b$.

Proof. $x_1=(a\smile x_1)\frown(x_1\smile x_2)=(a\smile x_1)\frown(a\smile b)=a\smile x_1$ by ax_1x_2, Theorem 4, and hence $x_1\geqq a$. On the other hand, $x_1=(a\frown x_1)\smile(x_1\frown x_2)$ $=(a\frown x_1)\smile(a\frown b)=a\frown x_1$ by ax_1x_2, Theorem 4, and hence $x_1\leqq a$. Accordingly we have $x_1=a$. Since x_1x_2b from ax_2b, and hence we have $x_2=b$ similarly.

Corollary 4.2. Let L be a complemented distributive lattice with $I, 0$. If we take a, b of L such that $a\smile b=I$, $a\frown b=0$, then we have $B(a,b)=B(I,0)=L$.

Proof is evident from Theorem 4.

Now we consider the structure of $B(a,b)$ in case there is a maximal chain between a and $a\frown b$. Suppose that a_2 covers $a_1=a_2\frown b$, and a_2xb, then we get $a_2\geqq a_2\frown x\geqq a_1$ by Lemma 2, hence we have either $a_2=a_2\frown x$ or $a_2\frown x=a_1$ from $a_2\succ a_1$. In the first case we have $a_2\smile b\geqq x\geqq a_2$. In the second case $a_1\smile(x\frown b)=x$ from $(a_2\frown x)\smile(x\frown b)=x$, hence $x\frown b=x$ since $x, b\geqq a_1$.

Define $C_{a_i}=\{x\mid a_i\leqq x\leqq a_i\smile b\}$; then we have $B(a_2,b)=\sum_{i=1}^{2}C_{a_i}$. In the same way, if there is a maximal chain between a and $a\frown b$ such that $a=a_n\succ a_{n-1}\succ\cdots\succ a_1=a\frown b$, then we obtain $B(a,b)=\sum_{i=1}^{n}C_{a_i}$.

Theorem 5. If a lattice is generated by the two maximal chains $\{a_n\}$, $\{b_m\}$ *such that*

$$a=a_n\succ a_{n-1}\succ\cdots\succ a_1=a\frown b,$$
$$b=b_m\succ b_{m-1}\succ\cdots\succ b_1=a\frown b,$$

then $B(a,b)$ *consists of* mn *lattice points.*

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