72. On a Theorem of Ugaheri

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1. In his paper [6], Ugaheri investigated the potentials in m-dimensional euclidean space R^m, with respect to a kernel function $\Phi(r)$ which is positive, continuous and monotonously decreasing at $r>0$, and he proved that these potentials satisfy the following maximum principle: if the potential of a positive measure with compact carrier is not greater than M on the carrier of the measure, then it is $\leqq kM$ everywhere in R^m, where k is an absolute constant depending only upon the space R^m. Furthermore, using this maximum principle, he proved Evans-Vasilesco's theorem for the potentials with respect to $\Phi(r)$.

In the present note we shall consider the potentials in a locally compact space Ω and prove that the generalized form of the maximum principle of Ugaheri is equivalent to the continuity principle with certain additional condition, where the continuity principle means that, if the potential of a positive measure, considered as a function on the carrier of the measure, is continuous, then it is also continuous in the whole space.

2. We assume that there is given a real-valued function $\Phi(p, q)$, defined in the product space $\Omega\times\Omega$ and satisfying the following conditions:

1° $\Phi(p, q)$ is positive, continuous in (p, q) except for the diagonal set of $\Omega\times\Omega$ and symmetric, that is, $\Phi(p, q)=\Phi(q, p)$:

2° At every point p of Ω, $\lim_{q\to p}\Phi(p, q)=+\infty$ and, in case Ω is not compact, $\lim_{q\to\omega}\Phi(p, q)=0$, where ω is the Alexandroff point.

The potential $U^\mu(p)$ of a positive measure μ is defined by the Radon-Stieltjes integral

$$U^\mu(p)=\int_\Omega \Phi(p, q)\, d\mu(q).$$

We assume the following condition:

3° $\Phi(p, q)$ satisfies the energy principle in the sense of Ninomiya [4].

Let K be a compact subset of Ω, and denote by $\mathfrak{M}_K^1$ the family of all positive measures on K which are of total measure 1. We put

$$W(K)=\inf_{\mu\in\mathfrak{M}_K^1}\int U^\mu\, d\mu,$$

and define the capacity $c(K)$ of K as follows: when $W(K)$ is finite, $c(K)=1/W(K)$, and when $W(K)=+\infty$, $c(K)=0$. For every subset A of Ω we define interior capacity $c_i(A)$ by $\sup c(K)$, taken with respect to compact $K\subset A$. In what follows we assume the following condition:

4° Everv open set $G\neq\phi$ is of positive interior capacity.

We say that a property holds nearly everywhere in Ω if it holds at each point of Ω except at the points of a set of interior capacity zero (cf. Cartan [2] and Choquet [3]).

It is well known that, if B_n $(n=1, 2, \cdots)$ are Borel sets and B is the union of B_n, then $c_i(B)\leqq\sum_{n=1}^{\infty}c_i(B_n)$.

We denote by $\mathfrak{E}$ the family of all positive measures of finite energy. By assumption 3°, we can define a strong topology in $\mathfrak{E}$ by means of the square root of energy integrals. Modifying Cartan's proof [2], we can easily show

Lemma 1. *Let μ and ν be measures of $\mathfrak{E}$, ε be a positive number and B be a set of points p of Ω such that $U^{\mu}(p)-U^{\nu}(p)>\varepsilon$. Then $c_i(B)\leqq\frac{1}{\varepsilon^2}\|\mu-\nu\|^2$, where $\|\mu-\nu\|^2$ is the energy of $\mu-\nu$.*

3. We generalize the maximum principle of Ugaheri as follows: let K be a compact subset of Ω; for any measure μ of $\mathfrak{E}_K$, $U^{\mu}(p)\leqq k\sup_{q\in K}U^{\mu}(q)$ at every point p of Ω, where k is a constant depending only upon K and $\mathfrak{E}_K$ means the subfamily of $\mathfrak{E}$, each measure of which has its carrier in K.

We shall prove

Theorem. *The generalized maximum principle of Ugaheri is equivalent to the continuity principle and the following condition:*

(*) *if the potential of a positive measure with compact carrier is bounded on the carrier of the measure, then it is also bounded in Ω.*

4. In this section we shall prove that the continuity principle and the condition (*) are sufficient for the generalized maximum principle of Ugaheri. At first we prove some propositions (Lemmas 2, 3 and 4) under the assumption of the continuity principle.

Lemma 2. *Let K be a compact set and μ be an element of $\mathfrak{E}_K$. Then we can find a sequence $\{\mu_n\}$ of positive measures with the following properties: 1° $\{\mu_n\}$ converges to μ both vaguely and strongly, 2° the potentials U^{μ_n} are all continuous in Ω, and 3° $\{U^{\mu_n}(p)\}$ converges increasingly to $U^{\mu}(p)$ at every point p of Ω.*

Proof. As $U^{\mu}(p)$ is a measurable function, we can take a sequence $\{K_n\}$ $(n=1, 2, \cdots)$ of compact sets such that $\mu(\Omega-K_n)<1/n$ and U^{μ} is continuous on each K_n. We may suppose that $\{K_n\}$ is increasing. Let μ_n be the restriction of μ to K_n. Then U^{μ_n} is continuous on K_n,*) whence, by the continuity principle, U^{μ_n} is continuous in Ω. As $\{K_n\}$ is increasing, $\{U^{\mu_n}\}$ is also increasing, and $\lim_n U^{\mu_n}(p) \leqq U^{\mu}(p)$ at every point p of Ω. Let $f(p)$ be a non-negative continuous function with compact carrier and M be the maximum value of $f(p)$. Then

$$0 \leqq \int f\,d\mu - \int f\,d\mu_n = \int_{\Omega - K_n} f\,d\mu_n \leqq M\mu(\Omega - K_n) < M/n.$$

Therefore, $\{\mu_n\}$ converges vaguely to μ and $U^{\mu}(p) \leqq \lim_n U^{\mu_n}(p)$. Thus $\{U^{\mu_n}(p)\}$ converges increasingly to $U^{\mu}(p)$ at every point p of Ω.

On the other hand, we have

$$\int U^{\mu}\,d\mu = \lim_n \int U^{\mu_n}\,d\mu \geqq \lim_n \int U^{\mu_n}\,d\mu_n$$

and

$$0 \leqq \overline{\lim_n}\left\{\int U^{\mu}\,d\mu - 2\int U^{\mu_n}\,d\mu + \int U^{\mu_n}\,d\mu_n\right\}$$
$$= \lim_n \int U^{\mu_n}\,d\mu_n - \int U^{\mu}\,d\mu \leqq 0,$$

whence we have $\lim_n \|\mu - \mu_n\| = 0$, which proves the strong convergence of $\{\mu_n\}$. q.e.d.

We call each μ_n of the above lemma the *smoothed measure* of μ. Using the smoothed measures we can easily prove the following lemma, which was proved by Ohtsuka [5].

Lemma 3. *$\mathfrak{E}_K$ is complete in the strong topology.*

Next we shall prove

Lemma 4. *Let μ and μ_n $(n=1, 2, \cdots)$ be measures of $\mathfrak{E}_K$. If $\{\mu_n\}$ converges strongly to μ, then it converges vaguely to μ.*

Proof. Contrary to the assertion, we suppose that $\{\mu_n\}$ does not converge vaguely to μ. Then there exist a continuous function $f(p)$ and a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $\left\{\int f\,d\mu_{n_k}\right\}$ converges to $\alpha \neq \int f\,d\mu$. As $\{\mu_{n_k}\}$ converges strongly to μ, $\|\mu_{n_k}\|$ $(k=1, 2, \cdots)$ are bounded from above, i.e., $\|\mu_{n_k}\| < M < +\infty$. Hence $\{\mu_{n_k}\}$ is bounded in the space $\mathfrak{M}(\Omega)$ of all measures in Ω, and a subsequence $\{\mu^{(j)}\}$ of $\{\mu_{n_k}\}$ converges vaguely to a positive measure ν (cf. [1], p. 62). Hence, for the above function $f(p)$, $\lim \int f\,d\mu^{(j)} = \int f\,d\nu$ and $\int f\,d\nu \neq \int f\,d\mu$.

On the other hand, by Lemma 2, there exists a smoothed measure $\bar{\nu}$ of ν, for an arbitrary ε, such that $\|\nu - \bar{\nu}\| < \varepsilon/M$ and $U^{\bar{\nu}}$ is continuous in Ω. Then, for each j, $\left|\int U^{\nu}\,d\mu^{(j)} - \int U^{\bar{\nu}}\,d\mu^{(j)}\right| \leqq \|\nu - \bar{\nu}\|$

*) Prof. Deny remarked to the author that this is an immediate consequence of the following proposition: if the sum of two lower semi-continuous functions is continuous, then each of them is continuous.

$\|\mu^{(j)}\|<\varepsilon$. Letting j tend to infinity, we have $\left|\int U^{\nu}\,d\mu-\int U^{\bar{\nu}}\,d\nu\right|\leqq\varepsilon$, since $\int U^{\nu}\,d\mu^{(j)}$ and $\int U^{\bar{\nu}}\,d\mu^{(j)}$ converge to $\int U^{\nu}\,d\mu$ and $\int U^{\bar{\nu}}\,d\nu$, respectively. Then we let ε tend to zero to get $\int U^{\nu}\,d\mu=\int U^{\nu}\,d\nu$. In the same way, we get $\int U^{\mu}\,d\mu=\int U^{\mu}\,d\nu$. Thus $\|\mu-\nu\|=0$ and so $\mu=\nu$. Hence, for the above function $f(p)$, $\int f\,d\mu=\int f\,d\nu$, which contradicts our supposition.

Now we shall prove that the continuity principle and the condition (*) imply the generalized maximum principle of Ugaheri.

For a compact set K, we put

$$\mathfrak{E}^*(K)=\{\mu\in\mathfrak{E}_K:\sup_{p\in K}U^{\mu}(p)<+\infty\}.$$

We can define a uniform topology in $\mathfrak{E}^*(K)$ as follows: Let μ_0 be an element of $\mathfrak{E}^*(K)$ and ε be a positive number. The ε-neighborhood $N(\mu_0,\varepsilon)$ of μ_0 is the set of elements μ of $\mathfrak{E}^*(K)$ such that $|U^{\mu}-U^{\mu_0}|<\varepsilon$ nearly everywhere on K. Then $\mathfrak{E}^*(K)$ is separable. To show this, we may prove that $\bigcap_{\varepsilon>0}N(\mu_0,\varepsilon)=\{\mu_0\}$, that is, if $U^{\mu_0}=U^{\nu}$ nearly everywhere on K, then $\mu_0=\nu$. In fact, as $U^{\mu_0}=U^{\nu}$ nearly everywhere on K, it is easily seen that

$$\int U^{\mu_0}\,d\mu_0=\int U^{\nu}\,d\mu_0=\int U^{\mu_0}\,d\nu=\int U^{\nu}\,d\nu,$$

whence we have $\mu_0=\nu$. Since every element of $\mathfrak{E}^*(K)$ has a countable base of neighborhoods, $\mathfrak{E}^*(K)$ is metrisable.

Next we shall prove that $\mathfrak{E}^*(K)$ is complete in our topology. In order to prove this, first we shall show that any Cauchy sequence in $\mathfrak{E}^*(K)$ is also a Cauchy sequence in $\mathfrak{E}_K$. Let $\{\mu_n\}$ be a Cauchy sequence in $\mathfrak{E}^*(K)$. Then, for an arbitrary $\varepsilon>0$, there exists an integer n_0 such that $|U^{\mu_n}(p)-U^{\mu_m}(p)|<\varepsilon$ nearly everywhere on K for all $n, m\geqq n_0$. Particularly, $U^{\mu_n}(p)<U^{\mu_{n_0}}(p)+\varepsilon$ nearly everywhere on K for all $n\geqq n_0$, and hence $U^{\mu_n}(p)<\sup_{p\in K}U^{\mu_{n_0}}(p)+\varepsilon$ nearly everywhere on K for all $n\geqq n_0$. Then, by assumption 4°, $U^{\mu_n}(p)<\sup_{p\in K}U^{\mu_{n_0}}(p)+\varepsilon$ on K for all $n\geqq n_0$. Hence $U^{\mu_n}(p)$ $(n=1,2,\cdots)$ are bounded from above by a finite number M on K, that is, $\int\Phi(p,q)\,d\mu_n(q)<M<+\infty$ on K. Since $\Phi(p,q)>a>0$ on $K\times K$, $\int d\mu_n(p)<M/a$. Then we have $\|\mu_n-\mu_m\|^2=\int(U^{\mu_n}-U^{\mu_m})\,d(\mu_n-\mu_m)\leqq\int|U^{\mu_n}-U^{\mu_m}|\,d\mu_n+\int|U^{\mu_n}-U^{\mu_m}|\,d\mu_m<\varepsilon\left(\int d\mu_n+\int d\mu_m\right)<2\varepsilon M/a$. This shows that $\{\mu_n\}$ is a Cauchy sequence in $\mathfrak{E}_K$.

Now we shall prove the completeness of $\mathfrak{E}^*(K)$. Let $\{\mu_n\}$ be a Cauchy sequence in $\mathfrak{E}^*(K)$. Then $\{\mu_n\}$ is a Cauchy sequence in $\mathfrak{E}_K$, as shown above. Then, by virtue of the completeness of $\mathfrak{E}_K$, there exists a measure μ of $\mathfrak{E}_K$ such that $\{\mu_n\}$ converges strongly to μ in $\mathfrak{E}_K$. By Lemma 4, $\{\mu_n\}$ converges vaguely to μ, whence $U^{\mu}\leqq \varliminf_n U^{\mu_n}$ and μ belongs to $\mathfrak{E}^*(K)$. Next we shall prove that $\{\mu_n\}$ converges to μ in $\mathfrak{E}^*(K)$. Let $\{\varepsilon_j\}$ be a sequence of positive numbers such that $\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_j>\cdots$ and $\lim_j \varepsilon_j=0$. By our assumption, there exists an integer n_0 for each j such that, for every $n, m\geqq n_0$, $|U^{\mu_n}(p)-U^{\mu_m}(p)|<\varepsilon_j$ except at the points of a set $A(\varepsilon_j)$ whose interior capacity is zero. Put $A=\bigcup_{j=1}^{\infty} A(\varepsilon_j)$. Then A is of interior capacity zero and $|U^{\mu_n}(p)-U^{\mu_m}(p)|<\varepsilon_j$ in $K-A$ for every $n, m\geqq n_0$. Hence the limiting function $U(p)=\lim_n U^{\mu_n}(p)$ exists in $K-A$. We can prove that $U(p)=U^{\mu}(p)$ holds nearly everywhere in $K-A$. In fact, for a positive number ε, we set

$$B_\varepsilon=\{p\in K-A: U(p)>U^{\mu}(p)+\varepsilon\}.$$

For this number ε, there exists n_0 such that, for every $n\geqq n_0$, $U^{\mu_n}(p)>U(p)-\varepsilon/2$ in $K-A$. Therefore, $B_\varepsilon^n=\{p\in K-A: U^{\mu_n}(p)> U^{\mu}(p)+\varepsilon/2\}$ contains B_ε for every $n\geqq n_0$. Hence we have $c_i(B_\varepsilon)=0$. On the other hand, we have

$$B=\{p\in K-A: U(p)>U^{\mu}(p)\}=\bigcup_{j=1}^{\infty} B_{1/j}$$

and $c_i(B)\leqq \sum_{j=1}^{\infty} c_i(B_{1/j})=0$. Therefore, $U\leqq U^{\mu}$ nearly everywhere in $K-A$. Hence $U=U^{\mu}$ nearly everywhere in $K-A$. Thus we have proved that $\{\mu_n\}$ converges to μ in $\mathfrak{E}^*(K)$ and $\mathfrak{E}^*(K)$ is complete.

Now, for every positive integer k, we put

$$\mathfrak{E}_k=\{\mu\in\mathfrak{E}^*(K): U^{\mu}(p)\leqq k\sup_{q\in K} U^{\mu}(q) \text{ in } \Omega\}.$$

Then, by assumption 4°, $\mathfrak{E}_k$ is closed in $\mathfrak{E}^*(K)$ and $\mathfrak{E}^*(K)=\bigcup_{k=1}^{\infty}\mathfrak{E}_k$ by condition (*). As $\mathfrak{E}^*(K)$ is a complete metrisable space, we can find by Baire's theorem an integer k_0, $\mu_0\in\mathfrak{E}_{k_0}$ and a neighborhood $N(\mu_0)$ of μ_0 such that $N(\mu_0)\subset\mathfrak{E}_{k_0}$. We may assume that $N(\mu_0)$ is written as follows:

$$N(\mu_0)=\{\mu\in\mathfrak{E}^*(K): |U^{\mu}-U^{\mu_0}|<\varepsilon_0 \text{ nearly everywhere on } K\}.$$

Put $M_{\mu_0}=\sup_{p\in K} U^{\mu_0}(p)$ and $M_0=M_{\mu_0}+\varepsilon_0$. For every $\nu\in\mathfrak{E}^*(K)$, $\mu_0+\dfrac{\varepsilon_0}{2\sup_{p\in K} U^{\nu}(p)}\nu$ belongs to $N(\mu_0)$ and $\dfrac{\varepsilon_0}{2\sup_{p\in K} U^{\nu}(p)}U^{\nu}(p)\leqq k_0M_0$, whence

$$U^{\nu}(p)\leqq \frac{2k_0M_0}{\varepsilon_0}\sup_{q\in K} U^{\nu}(q) \text{ in } \Omega.$$

Thus we have proved that there exists a constant $k=\frac{2k_0M_0}{\varepsilon_0}$ for a compact set K such that, for every $\nu \in \mathfrak{E}_K$,

$$U^\nu(p) \leqq k \sup_{q \in K} U^\nu(q) \text{ in } \Omega.$$

5. Conversely we shall prove that the continuity principle follows from the generalized maximum principle of Ugaheri.

Suppose that the potential U^μ of a positive measure μ is continuous on the compact carrier K of μ. For any point p_0 of K, we can take a sequence $\{L_n\}$ of compact sets, converging decreasingly to p_0. Then, since U^μ is continuous on K, the potentials U^{μ_n} of the restrictions μ_n of μ to L_n are continuous on K,*) and they converge uniformly to zero. Using this and the generalized maximum principle of Ugaheri, we can easily show that $\overline{\lim_{p \to p_0}} U^\mu(p) \leqq U^\mu(p_0)$. From this, the continuity of U^μ in Ω follows immediately.

Thus we have proved our theorem completely.

References

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