24. *Uniform Convergence of Fourier Series.* VI

By Masako SATÔ
Mathematical Institute, Tokyo Metropolitan University, Tokyo
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6. Furthermore we can improve Theorem 6, in the following form:

Theorem 7. *If*

$$(1)\qquad \int_0^{|h|}(f(x+u)-f(x))du=o(|h|),\ \textit{as}\ h\to 0$$

for a fixed x, and

$$(2)\qquad \frac{1}{h}\int_0^h(f(t+u)-f(t-u))du=o\Big(1\Big/\log\frac{1}{h}\Big),\ \textit{as}\ h\to 0$$

uniformly for all t, then the Fourier series of $f(t)$ converges at x.

In other words the condition in Theorem 6

$$\int_0^{|h|}|f(x+u)-f(x)|du=o(|h|)$$

is replaced by (1).

Proof. We put

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}dt+o(1)=\frac{1}{\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^\pi\Big]+o(1)$$
$$=\frac{1}{\pi}[I+J]+o(1).$$

Then by integration by parts

$$I=\int_0^{\pi/n}\Phi_x(t)\Big(\frac{\sin nt}{t^2}-\frac{n\cos nt}{t}\Big)dt,$$

and hence, on account of (2), the absolute value of I is not greater than

$$2n\int_0^{\pi/n}|\Phi_x(t)|\frac{dt}{t}=o\Big(n\int_0^{\pi/n}dt\Big)=o(1),$$

where $\Phi_x(t)=\int_0^t\varphi_x(u)du=o(t)$ as $n\to\infty$ $(0\leqq t\leqq\pi/n)$.

In order to evaluate J we now put (cf. [4])

$$J=\int_{\pi/n}^\pi\varphi_x(t)\frac{\sin nt}{t}dt=J_1-J_2,$$

where

$$J_1=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt,$$

$$J_2=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\varphi_x(t+(2k-1)\pi/n)\Big(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k-1)\pi/n}\Big)\sin nt\,dt,$$

and further we divide J_1 into two parts as follows:

$$J_1=\sum_{k=1}^{(n-1)/2}\left[\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{2k\pi/n}\sin nt\,dt\right.$$
$$\left.-\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{(t+2k\pi/n)\cdot 2k\pi/n}t\sin nt\,dt\right]$$
$$=J_{11}-J_{12}.$$

We write

$$J_{11}=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}\left[\int_0^{\pi/n}[f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)]\sin nt\,dt\right.$$
$$\left.+\int_0^{\pi/1}[f(x-t-2k\pi/n)-f(x-t-(2k-1)\pi/n)]\sin nt\,dt\right]$$
$$=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}[J_{11}^1+J_{11}^2],$$

then

$$J_{11}^1=\int_0^{\pi/2n}[f(x+2k\pi/n+t)-f(x+(2k-1)\pi/n+t)]\sin nt\,dt$$
$$+\int_0^{\pi/2n}[f(x+2k\pi/n+(\pi/n-t))-f(x+(2k-1)\pi/n+(\pi/n-t))]\sin nt\,dt$$
$$=\int_0^{\pi/2n}[f(x+2k\pi/n+t)-f(x+2k\pi/n-t)]\sin nt\,dt$$
$$-\int_0^{\pi/2n}[f(x+(2k-1)\pi/n+t)-f(x+(2k+1)\pi/n-t)]\sin nt\,dt$$
$$=\int_0^{\pi/2n}[f(\xi+t)-f(\xi-t)]\sin nt\,dt-\int_{\pi/2n}^{\pi/n}[f(\xi+\tau)-f(\xi-\tau)]\sin n\tau d\tau$$
$$=2\int_0^{\pi/2n}[f(\xi+t)-f(\xi-t)]\sin nt\,dt-\int_0^{\pi/n}[f(\xi+t)-f(\xi-t)]\sin nt\,dt,$$

where $\xi=x+2k\pi/n$, $\tau=t-\pi/n$. By integration by parts and (2)

$$\int_0^{\pi/2n}(f(\xi+t)-f(\xi-t))\sin nt\,dt=\left[\sin nt\int_0^t(f(\xi+u)-f(\xi-u))\,du\right]_0^{\pi/2n}$$
$$-n\int_0^{\pi/2n}\cos nt\,dt\int_0^t(f(\xi+u)-f(\xi-u))\,du$$
$$=o(1/n\log n)+o\left(n\int_0^{\pi/2n}\frac{t}{\log 1/t}\,dt\right)=o(1/n\log n),$$

and similarly

$$\int_0^{\pi/n}(f(\xi+t)-f(\xi-t))\sin nt\,dt=o(1/n\log n).$$

Hence we have

$$\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{11}^1=\sum_{k=1}^{(n-1)/2}\frac{n}{k}o\left(\frac{1}{n\log n}\right)=o\left(\frac{1}{\log n}\sum_{k=1}^{(n-1)/2}\frac{1}{k}\right)=o(1),$$

and quite similarly $\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{11}^2=o(1)$, thus we get $J_{11}=o(1)$.

On the other hand we put

$$J_{12}=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}\Big[\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{t+2k\pi/n}t\sin nt\,dt$$
$$+\int_0^{\pi/n}\frac{f(x-t-2k\pi/n)-f(x-t-(2k-1)\pi/n)}{t+2k\pi/n}t\sin nt\,dt\Big]$$
$$=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}[J_{12}^1+J_{12}^2],$$

then by integration by parts and (2), we have

$$J_{12}^1=-\int_0^{\pi/n}F_x(t)\frac{(2k\pi/n)\sin nt+nt^2\cos nt+2k\pi t\cos nt}{(t+2k\pi/n)^2}dt$$

and hence

$$\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{12}^1=\sum_{k=1}^{(n-1)/2}\Big(\frac{n}{k}\Big)^3\int_0^{\pi/n}o\Big(\frac{1}{n\log n}\Big)(nt^2+4tk)\,dt$$
$$=o\Big(\frac{n^3}{n\log n}\sum_{k=1}^{n}\frac{1}{k^3}\Big(\frac{1}{n^3}+\frac{k}{n^2}\Big)\Big)=o\Big(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k^2}\Big)=o(1),$$

where $F_x(t)=\int_0^t[f(x+u+2k\pi/n)-f(x+u+(2k-1)\pi/n)]\,du=o(1/n\log n)$ uniformly for x and k as $n\to\infty$ $(0\leqq t\leqq\pi/n)$. In the same way we get $\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{12}^2=o(1)$, thus we have $J_{12}=o(1)$.

Finally we shall prove $J_2=o(1)$. By Abel's lemma

$$J_2=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\sum_{j=k}^{n}\Big(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\Big)$$
$$\cdot(\varphi_x(t+(2k-1)\pi/n)-\varphi_x(t+(2k-3)\pi/n))\sin nt\,dt$$
$$+\int_0^{\pi/n}\sum_{j=1}^{n}\Big(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\Big)\varphi_x(t+\pi/n)\sin nt\,dt$$
$$=J_{21}+J_{22},$$

say. Then by integration by parts

$$J_{21}=-\sum_{k=1}^{(n-1)/2}\frac{\pi}{n}\int_0^{\pi/n}\Big[\int_0^t(\varphi_x(u+(2k-1)\pi/n)-\varphi_x(u+(2k-3)\pi/n))\,du\Big]$$
$$\cdot\sum_{j=k}^{n}\Big(\frac{n\cos nt}{(t+2j\pi/n)(t+(2j-1)\pi/n)}-\frac{\sin nt\,(2t+(4j-1)\pi/n)}{(t+2j\pi/n)^2(t+(2j-1)\pi/n)^2}\Big)dt,$$

whence

$$J_{21}=\sum_{k=1}^{(n-1)/2}\frac{\pi}{n}\int_0^{\pi/n}\sum_{j=k}^{n}\Big(\frac{n^3}{j^2}+\frac{n^3}{j^3}\Big)o\Big(\frac{1}{n\log n}\Big)dt$$
$$=o\Big(\sum_{k=1}^{n}\frac{1}{\log n}\sum_{j=k}^{n}\frac{1}{j^2}\Big)=o\Big(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\Big)=o(1),$$

by condition (2). Furthermore, we have also by integration by parts

$$J_{22}=-\frac{\pi}{n}\int_0^{\pi/n}\Big[\int_0^t\varphi_x(u+\pi/n)\,du\Big]$$

$$\cdot\sum_{j=1}^{n}\left(\frac{n\cos nt}{(t+2j\pi/n)(t+(2j-1)\pi/n)}-\frac{\sin nt\,(2t+(4j-1)\pi/n)}{(t+2j\pi/n)^2(t+(2j-1)\pi/n)^2}\right)dt$$

and applying condition (1)

$$|J_{22}|\leqq A\frac{1}{n}\sum_{j=1}^{n}\frac{n^3}{j^2}\int_0^{\pi/n}\left|\int_0^t[f(x+u+\pi/n)-f(x)]\,du\right.$$

$$\left.+\int_0^t[f(x-u-\pi/n)-f(x)]\,du\right|dt$$

$$\leqq An^2\int_0^{\pi/n}\left|\left[\int_0^{t+\pi/n}+\int_0^{\pi/n}+\int_{-\pi/n}^0+\int_{-\pi/n-t}^0\right](f(x+u)-f(x))\,du\right|dt$$

$$=o(1),$$

where A is an absolute constant. Thus the theorem is proved.

7. We can prove the following theorems analogously as Theorems 3, 4 and 5.

Theorem 8. *Let* $0<\alpha<1$. *If*

(1) $$\int_0^{|h|}(f(x+u)-f(x))\,du=o(|h|),\quad as\quad h\to 0,$$

for a fixed x, *and*

$$\frac{1}{h}\int_0^h(f(t+u)-f(t-u))du=o\left(1\Big/\left(\log\frac{1}{h}\right)^{\alpha}\right),\quad as\quad h\to 0$$

uniformly for all t, *and further nth Fourier coefficients of* $f(t)$ *are of order* $O(e^{(\log n)^{\alpha}}/n)$, *then the Fourier series of* $f(t)$ *converges at* x.

Theorem 9. *Let* $\alpha>1$. *If (1) holds and*

$$\frac{1}{h}\int_0^h(f(t+u)-f(t-u))\,du=o\left(1\Big/\left(\log\log\frac{1}{h}\right)^{\alpha}\right),\quad as\quad h\to 0$$

uniformly for all t *and the nth Fourier coefficients of* $f(t)$ *are of order* $O(e^{(\log\log n)^{\alpha}}/n)$, *then the Fourier series of* $f(t)$ *converges at* x.

If $\alpha=1$, *then the conclusion holds when* $O(e^{(\log\log n)^{\alpha}}/n)$ *in the last condition is replaced by* $O((\log n)^{\gamma}/n)$ $(\gamma>0)$.

Theorem 10. *If (1) holds and*

$$\frac{1}{h}\int_0^h(f(t+u)-f(t-u))\,du=o\left(1\Big/\psi\left(\frac{1}{h}\right)\right),\quad as\quad h\to 0$$

uniformly for all t *and if* $f(t)$ *is of class* $\phi(n)$ *then the Fourier series of* $f(t)$ *converges at* x, *where* $\phi(n)=O(n)$, $\psi(n)=\log(n\theta(n)/\phi(n))$ *and* $\theta(n)$ *are monotone increasing to infinity as* $n\to\infty$.

8. R. Salem [1] proved the following theorem concerning the partial sum of Fourier series.

Theorem 11. *If* $f(x)\varepsilon L$ *and*

(1) $$\frac{1}{h}\int_0^h(f(t+u)-f(t-u))du=O\left(1\Big/\log\frac{1}{h}\right),\quad as\quad h\to 0$$

uniformly for all t, *then*

(2) $$|s_n(x)|<g(x)\qquad(n=1,2,\cdots)$$

where $g(x)\varepsilon L^{\mu}$ $(0<\mu<1)$.

Further if $f(x)\varepsilon L^{r}$ $(r>1)$ *and* (1) *holds then* (2) *is true for* $g(x)\varepsilon L^{r}$, *and if* $f(x)\log^{+}|f(x)|\varepsilon L$ *and* (1) *holds, then* (2) *is true for* $g(x)\varepsilon L$.

From the proof of R. Salem, we can see that

$$|s_n(x)|\leqq A \max_{\alpha\leqq x\leqq\beta}\int_{\alpha}^{\beta}|f(t)|\,dt+O(1),$$

from which the above theorem is deduced by the maximal theorem [6]. We shall prove the following slight generalization by the method used above.

Theorem 12. *If* $f(x)\varepsilon L$ *and* (1) *holds, then*

(3) $$|s_n(x)|\leqq 16\ \theta(x,f)+O(1),$$

where

$$\theta(x,f)=\max_{\alpha\leqq x\leqq\beta}\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)\,dt\right|.$$

Proof. We put $\phi_x(t)=f(x+t)-f(x-t)$ and

$$s_n(x)=\frac{1}{\pi}\int_0^{\pi}\phi_x(t)\frac{\sin nt}{t}dt+o(1)=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]+o(1)$$
$$=\frac{1}{\pi}[I+J]+o(1).$$

Then by integration by parts

$$I=\int_0^{\pi/n}\left(\frac{\sin nt}{t^2}-\frac{n\cos nt}{t}\right)\int_0^{t}\phi_x(u)\,du\,dt,$$

hence we have

$$|I|\leqq 2n\int_0^{\pi/n}\left|\frac{1}{t}\int_0^{t}\phi_x(u)\,du\right|dt\leqq 4n\,\theta(x,f)\int_0^{\pi/n}dt=4\pi\ \theta(x,f).$$

Hence it is sufficient to prove that $J=O(1)+12\pi\,\theta(x,f)$. As in the proof of Theorem 6 we get $J_{11}=O(1)$, $J_{12}=o(1)$ and $J_{21}=O(1)$. Thus it remains only to show that $|J_{22}|\leqq 12\pi\,\theta(x,f)$. Integrating by parts we have

$$J_{22}=-\frac{\pi}{n}\int_0^{\pi/n}\int_0^{t}\phi_x(u+\pi/n)\,du$$
$$\cdot\sum_{j=1}^{n}\left(\frac{n\cos nt}{(t+2j\pi/n)(t+(2j-1)\pi/n)}-\frac{\sin nt\,(2t+(4j-1)\pi/n)}{(t+2j\pi/n)^2\,(t+(2j-1)\pi/n)^2}\right)dt$$

and then

$$|J_{22}|\leqq\frac{\pi}{n}\sum_{j=1}^{n}\left(\frac{n}{(2j-1)^2\pi^2/n^2}-\frac{2}{(2j-1)^3\pi^3/n^3}\right)\int_0^{\pi/n}\left|\int_0^{t}\phi_x(u+\pi/n)\,du\right|dt$$
$$\leqq\frac{\pi}{n}\frac{n^3}{\pi^2}\sum_{j=1}^{n}\left(\frac{1}{(2j-1)^2}+\frac{2}{(2j-1)^3\pi}\right)\int_0^{\pi/n}\left|\int_0^{t}\phi_x(u+\pi/n)\,du\right|dt$$
$$\leqq 6\pi\frac{n}{\pi}\theta(x,f)\sum_{j=1}^{n}\left(\frac{1}{(2j-1)^2}+\frac{2}{(2j-1)^3\pi}\right)\int_0^{\pi/n}dt\leqq 12\pi\ \theta(x,f).$$

Thus the result follows.

9. S. Izumi showed the author that Theorem 1 (iii) and Theorem 6 [4] are contained in his theorem [5]:

Theorem 13. *If*

$$\int_0^h |\varphi_x(u)|\, du = o(h), \quad as\ h\to 0$$

and

$$(1) \qquad n\int_0^{\pi/n} dt \left| \sum_{k=1}^{(n-1)/2} \int_{t+2k\pi/n}^{t+(2k+1)\pi/n} \frac{\varphi_x(u)-\varphi_x(u-\pi/n)}{u}\, du \right| = o(1)$$

as $n\to\infty$, *then the Fourier series of* $f(t)$ *converges at* x, *where* $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$.

For the proof it is sufficient to show that the condition (2) in Theorem 6 is implied by (1). This may be seen from the proof of Theorem 6.

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