120. Lacunary Fourier Series. II

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(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1955)

1. M. E. Noble [1] has proved the following

Theorem N. *If the Fourier series of $f(t)$ has a gap $0<|n-n_k|\leqq N_k$ such that*

$$\lim N_k/\log n_k=\infty$$

and $f(t)$ satisfies a Lipschitz condition of order α, where $\frac{1}{2}<\alpha<1$, in some interval $|x-x_0|\leqq\delta$. Then

$$\sum(|a_{n_k}|+|b_{n_k}|)<\infty,$$

where a_{n_k}, b_{n_k} are non-vanishing Fourier coefficients of $f(t)$.

As a continuation of the first paper [2] we treat absolute convergence of the Fourier series with a certain gap and satisfying some continuity condition at a point (Theorems 3 and 4).

We need following theorems and lemmas in [2].

Lemma 1. *Let (δ_m) be a sequence tending to zero and let $n=[4em/\delta_m]$. Then there exists a trigonometrical polynomial $T_n(x)$ of degree not exceeding n with constant term 1 such that*[1]

(i)	$\|T_n(x)\|\leqq A/\delta_m$,	*for all* x,
(ii)	$\|T_n(x)\|\leqq An/\delta_m e^m$,	*for* $\delta_m\leqq\|x\|\leqq\pi$,
(iii)	$\|T_n'(x)\|\leqq An/\delta_m$,	*for all* x,
(iv)	$\|T_n'(x)\|\leqq A(n^2/\delta_m e^m+1/x^2)$,	*for* $\lambda\delta_m\leqq\|x\|\leqq\pi$, $\lambda>1$,[2]
(v)	$\|T_n''(x)\|\leqq An^2/\delta_m$,	*for all* x,

where A denotes an absolute constant.

Theorem 1. *Let $0<\alpha<1$ and $0<\beta<min(1-\alpha,(2-\alpha)/3)$. If*

$$k^{2/(2-\alpha-3\beta)}<n_k<e^{2k/(2+\alpha+\beta)},$$

$$|n_{k\pm1}-n_k|>4ekn_k^\beta$$

and

$$\frac{1}{h^\beta}\int_0^\beta|f(t)-f(t\pm h)|\,dt=O(h^\alpha), \tag{1}$$

$$\frac{1}{\tau}\int_0^\tau|f(t)-f(t\pm h)|\,dt=O(1), \qquad \textit{unif. in } \tau\geqq h^\beta, \tag{2}$$

then

$$a_{n_k}=O(n_k^{-\alpha}),\quad b_{n_k}=O(n_k^{-\alpha}). \tag{3}$$

Lamma 2. *Let (δ_m) be a sequence tending to zero and let $n=[4me^{1-m\delta'm/\delta_m}/\delta_m]$. Then there exists a trigonometrical polynomial*

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1) A denotes an absolute constant which is not the same in different occurrences.

2) λ may be taken as near 1 as we like when m is sufficiently large.

$T_n(x)$ of degree not exceeding n with constant term 1, satisfying the conditions (i), (iii), (v) in Lemma 1 and

(ii′) $|T_n(x)| \leqq An/\delta_m e^{(1-m\delta'_m/\delta_m)(m-1)}$, $\delta_m \leqq |x| \leqq \pi$,

(iv′) $|T'_n(x)| \leqq A(n^2/\delta_m e^{(1-m\delta'_m/\delta_m)(m-1)} + 1/x^2)$, $\lambda\delta_m \leqq |x| \leqq \pi$, $\lambda > 1$.

Theorem 2. *Let $0<\alpha<1$, $0<\beta<(2-\alpha)/3$, and*

$$\gamma > 2/\min(1-\beta,\ 2-\alpha-3\beta)$$

(or especially $0<\beta<(1-\alpha)/2$ and $\gamma>2/(1-\beta)$). If the Fourier coefficients of $f(t)$ vanish except for $n=[k^\gamma]$ $(k=1,2,3,\dots)$ and the conditions (1) and (2) of Theorem 1 are satisfied, then (3) holds.

2. Theorem 3. *Let $1/2<a<\alpha<1$, $0<\beta<(2-\alpha)/3$, and $\beta/2<\alpha-a \leqq (2-\alpha-\beta)/4$. If*

$$k^{1/(2\alpha-2a-\beta)} < n_k < e^{2k/(2+\alpha+\beta)},$$
$$|n_{k\pm1}-n_k| > 4ekn_k^\beta$$

and

$$\frac{1}{h^\beta}\int_0^{h^\beta} |f(t)-f(t\pm h)|^2\,dt = O(h^{2a}) \quad \text{as } h\to 0, \tag{4}$$

$$\frac{1}{\tau}\int_0^{\tau} |f(t)-f(t\pm h)|^2\,dt = O(1) \qquad \text{unif. in } \tau>h^\beta \tag{5}$$

then

$$\sum(|a_{n_k}|+|b_{n_k}|)<\infty, \tag{6}$$

where a_{n_k}, b_{n_k} are the non-vanishing Fourier coefficients of $f(t)$.

Proof. Let $\delta_k = 1/n_k^\beta$ and choose a sequence $M_k=[4ek/\delta_k]$ and let $T_{M_k}(x)$ be the trigonometrical polynomial of Lemma 1. Let us put

$$g_k(x) = f\left(x+\frac{\pi}{4n_k}\right) - f\left(x-\frac{\pi}{4n_k}\right)$$

then

$$g_k(x) \sim \sum_0^\infty 2\sin\frac{n\pi}{4n_k}\cdot(b_n\cos nx - a_n\sin nx).$$

Then the nth Fourier coefficients α_n, β_n of $g_k(x)T_{M_k}(x)$ are given by

$$\alpha_{n_p} = 2\sin\frac{n_p\pi}{4n_k}b_{n_p}, \qquad \beta_{n_p} = -2\sin\frac{n_p\pi}{4n_k}a_{n_p}, \qquad (n_k \leqq n_p \leqq 2n_k).$$

On the other hand, by Theorem 1 we have

$$a_{n_k} = O(1/n_k^\alpha), \quad b_{n_k} = O(1/n_k^\alpha).$$

Since $\sum 1/n_k^{2\alpha} < \infty$, $f(x)$ belongs to the L^2-class. Thus we have

$$\begin{aligned}\frac{1}{2}\sum_{n_k}^{2n_k}(a_n^2+b_n^2) &\leqq \sum_{n_k}^{2n_k}(a_n^2+b_n^2)\sin^2\frac{n\pi}{4n_k} \\ &\leqq \frac{1}{4}\sum_{n_k}^{2n_k}(\alpha_n^2+\beta_n^2) \leqq \frac{1}{4\pi}\int_{-\pi}^{\pi} g_k^2(x)T_{M_k}^2(x)\,dx \\ &= \frac{1}{4\pi}\left[\int_0^\pi + \int_{-\pi}^0\right] g_k^2(x)T_{M_k}^2(x)\,dx = \frac{1}{4\pi}\left[I_1+I_2\right].\end{aligned}$$

By integration by parts

$$I_1=\left[T^2_{M_k}(x)\int_0^x g_k^2(t)\,dt\right]_0^\pi-2\int_0^\pi T_{M_k}(x)T'_{M_k}(x)\,dx\int_0^x g_k^2(t)dt$$
$$=I_{11}-2I_{12},$$

where

$$I_{11}=T^2_{M_k}(\pi)\int_0^\pi g_k^2(t)\,dt\leqq A\left(\frac{M_k}{\delta_k e^k}\right)^2=O\left(\frac{1}{n_k^{2a}}\right)$$

by Lemma 1, (ii), and for $\lambda>1$

$$I_{12}=\left[\int_0^{\lambda\delta_k}+\int_{\lambda\delta_k}^\pi\right]T_{M_k}(x)T'_{M_k}(x)dx\int_0^x g_k^2(t)dt$$
$$=I_{121}+I_{122}=O(1/n_k^{2a}).$$

For,

$$|I_{121}|\leqq\frac{AM_k}{\delta_k^2}\int_0^{\lambda\delta_k}dx\int_0^x g_k^2(t)dt$$
$$\leqq\frac{AM_k}{\delta_k}\int_0^{\lambda\delta_k}dx\left[\frac{1}{\delta_k}\int_0^{\lambda\delta_k}g_k^2(t)\,dt\right]$$
$$\leqq AM_k/n_k^{2a}\leqq A/n_k^{2a}.$$

By Lemma 1, (i), (iii) and condition (4) and

$$|I_{122}|\leqq\frac{AM_k}{\delta_k e^k}\int_{\lambda\delta_k}^\pi\left(\frac{M_k^2}{\delta_k e^k}+\frac{1}{x^2}\right)dx\int_0^x g_k^2(t)dt$$
$$\leqq\frac{AM_k^3}{\delta_k^2 e^{2k}}\int_{\lambda\delta_k}^\pi dx\int_0^x g_k^2(t)dt+\frac{AM_k}{\delta_k e^k}\int_{\lambda\delta_k}^\pi\frac{dx}{x}\left[\frac{1}{x}\int_0^x g_k^2(t)\,dt\right]$$
$$\leqq\frac{AM_k^3}{\delta_k^2 e^{2k}}+\frac{AM_k}{\delta_k e^k}\log\frac{1}{\delta_k}\leqq\frac{A}{n_k^{2a}}$$

by Lemma 1, (ii) and (iv).

Thus we have proved that

$$\sum_{n_k}^{2n_k}(a_n^2+b_n^2)=O(n_k^{-2a}).$$

Consequently

$$\sum_{2^m}^{2^{m+1}}(|a_n|+|b_n|)=O(2^{(\frac{1}{2}-a)m})$$

and then summing up both sides we get

$$\sum(|a_n|+|b_n|)<\infty.$$

Thus Theorem 3 is proved.

In a similar manner we can prove the following theorem, using Lemma 2 and Theorem 2.

Theorem 4. *Let* $1/2<a<\alpha<1$, $0<\beta<(1-\alpha)/2$, $\gamma>1/(2\alpha-2a-\beta)$, *and* $\beta/2<\alpha-a<(1+\beta)/4$.

If $n_k=[k^\gamma]$ $(k=1,2,3,\ldots)$, *and the conditions* (4) *and* (5) *are satisfied then* (6) *holds.*

References

[1] M. E. Noble: Coefficient properties of Fourier series with a gap condition, Math. Annalen, **128**, 55–62 (1954).

[2] M. Satô: Lacunary Fourier series. I, Proc. Japan Acad., **31**, 402–405 (1955).