95. Lacunary Fourier Series. I

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1. M. E. Noble [1] has proved the following

Theorem N. *If the Fourier series of* $f(t)$ *has a gap* $0<|n-n_k|\leqq N_k$ *such that*

$$\lim N_k/\log n_k=\infty$$

and $f(t)$ *satisfies a Lipschitz condition of order* $\alpha\,(0<\alpha<1)$ *in some interval* $|t-t_0|\leqq\delta$, *then*

$$a_{n_k}=O(1/n_k^{\alpha}),\quad b_{n_k}=O(1/n_k^{\alpha}),$$

where a_{n_k}, b_{n_k} *are non-vanishing Fourier coefficients of* $f(t)$.

In the present paper we treat the Fourier series with a certain gap and satisfying some continuity condition at a point, instead of in a small interval. Our theorems depend on the lemma (Lemma 1 in §2), which is due to M. E. Noble, except (iv) and (v).

We can also prove theorems concerning absolute convergence of Fourier series with the above-mentioned conditions, analogously to M. E. Noble [1]. These will be found in the second paper.

2. **Lemma 1.** *Let* (δ_m) *be a sequence tending to zero and let* $n=[4em/\delta_m]$. *Then there exists a trigonometrical polynomial* $T_n(x)$ *of degree not exceeding n with constant term* 1 *such that:*[1]

(i) $|T_n(x)|\leqq A/\delta_m$, *for all* x,

(ii) $|T_n(x)|\leqq An/\delta_m e^m$, $(\delta_m\leqq|x|\leqq\pi)$,

(iii) $|T_n'(x)|\leqq An/\delta_m$, *for all* x,

(iv) $|T_n'(x)|\leqq A(n^2/\delta_m e^m+1/x^2)$, $(\lambda\delta_m\leqq|x|\leqq\pi,\ \lambda>1)$[2]

(v) $|T_n''(x)|\leqq An^2/\delta_m$, *for all* x.

Proof. Let $E_m=(-\delta_m,\delta_m)$, and $C_m(x)$ be its characteristic function. We choose then $\tau_m=\delta_m/2m$ and construct a set of even function $h_i(x)$ $(i=0,1,2,\dots)$ defined by

$$h_0(x)=\frac{\pi}{\delta_m}C_m(x),$$

$$h_{i+1}(x)=\frac{1}{\tau_m}\int_x^{x+\tau_m}h_i(t)\,dt\qquad(i=0,1,2,\dots),$$

for $x\geqq0$ and $i\leqq m-1$.

It is easy to see that

$$h_m(x)=\begin{cases}0 & (\delta_m\leqq|x|\leqq\pi),\\ \pi/\delta_m & (|x|\leqq\delta_m/2),\end{cases}$$

1) A denotes an absolute constant which is not the same in different occurrences.

2) λ may be taken as near 1 as we like when m is sufficiently large.

and that it is monotone in the remaining intervals $[\delta_m/2, \delta_m]$ and $[-\delta_m, -\delta_m/2]$. Moreover it follows easily from the definition that

$$h_m^{(m-1)}(x)=O\Big(\Big(\frac{2}{\tau_m}\Big)^{m-1}\max|h_0(x)|\Big)=O\Big(\frac{(4m)^{m-1}}{\delta_m^m}\Big)$$

uniformly in x. If a_p and b_p are the pth Fourier coefficients of $h_m(x)$ we have, integrating $(m-1)$ times by parts,

$$\left.\begin{matrix}|a_p|\\|b_p|\end{matrix}\right\}\leqq\frac{1}{\pi p^{m-1}}\int_{-\pi}^{\pi}|h_m^{(m-1)}(x)|\,dx=O\Big(\frac{(4m)^{m-1}}{p^{m-1}\delta_m^m}\Big).$$

Consequently if $s_n(x)$ is the nth Fourier partial sum of $h_m(x)$,

$$|h_m(x)-s_n(x)|=O\Big(\frac{(4m)^{m-1}}{\delta_m^m}\sum_{p=n+1}^{\infty}\frac{1}{p^{m-1}}\Big)=O\Big(\frac{(4m)^{m-1}}{\delta_m^m n^{m-2}}\Big)$$

uniformly in $-\pi\leqq x\leqq\pi$. Taking $n=[4em/\delta_m]$ we obtain

$$|h_m(x)-s_n(x)|=O(n/\delta_m e^m)$$

which shows that the polynomial $s_n(x)$ satisfies (i) and (ii).

Further its constant term $a_0/2$ satisfies

$$\tfrac{1}{2}\leqq\tfrac{1}{2}a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}h_m(x)\,dx\leqq 1$$

and consequently the condition that the constant term is 1 can be satisfied by taking $T_n(x)=\lambda_n s_n(x)$ where $1\leqq\lambda_n\leqq 2$.

(iii) and (v) follow from (i) and (iii), respectively, by a famous inequality of Bernstein [2].

Finally we shall prove (iv). Since

$$T_n'(x)=\frac{2n}{\pi}\int_{-\pi}^{\pi}T_n(t+x)\ \sin nt\,K_{n-1}(t)\,dt,$$

where $K_n(t)$ is the Fejér kernel and $K_n(t)\leqq n$ $(0\leqq t\leqq\pi)$, $K_n(t)\leqq 1/nt^2$ $(1/n<t\leqq\pi)$ [2], we have

$$|T_n'(x)|\leqq An\Big[\int_{-\pi}^{-\pi/n}+\int_{-\pi/n}^{\pi/n}+\int_{\pi/n}^{-x-\delta_m}+\int_{-x-\delta_m}^{-x+\delta_m}+\int_{-x+\delta_m}^{\pi}\Big]|T_n(t+x)|K_{n-1}(t)dt$$

$$\leqq\frac{An}{\delta_m e^m}\Big[\int_{-\pi}^{-\pi/n}+\int_{\pi/n}^{-x-\delta_m}+\int_{-x+\delta_m}^{\pi}\Big]\frac{dt}{t^2}+\frac{An^3}{\delta_m e^m}\int_{-\pi/n}^{\pi/n}dt+\frac{A}{\delta_m}\int_{-x-\delta_m}^{-x+\delta_m}\frac{dt}{t^2}$$

$$\leqq\frac{An^2}{\delta_m e^m}+\frac{A}{x^2}.$$

Thus the lemma is completely proved.

Let $\delta(t)$ be a monotone decreasing sequence such that $\delta(t)\to 0$ as $t\to\infty$ and $\delta(t)$ is differentiable. We write $\delta(m)=\delta_m$ and $\delta'(m)=\delta_m'$.

In the estimation of $h_m(x)-s_n(x)$, the right side becomes minimum when

$$n=[4me^{1-m\delta_m'/\delta_m}/\delta_m].$$

For such n, we get

$$|h_m(x)-s_n(x)|=O(n/\delta_m e^{(1-m\delta_m'/\delta_m)(m-1)}).$$

Similarly to Lemma 1 we get the following

Lemma 2. *Let (δ_m) be a sequence tending to zero and let*

$n=[4me^{1-m\delta'_m/\delta_m}/\delta_m]$. *Then there exists a trigonometrical polynomial* $T_n(x)$ *of degree not exceeding* n *with constant term* 1, *satisfying the conditions* (*i*), (*iii*), (*v*), *Lemma 1, and*

(ii′) $\quad |T_n(x)|\leqq An/\delta_m e^{(1-m\delta'_m/\delta_m)(m-1)}, \qquad (\delta_m\leqq|x|\leqq\pi),$

(iv′) $\quad |T'_n(x)|\leqq A(n^2/\delta_m e^{(1-m\delta'_m/\delta_m)(m-1)}+1/x^2),$

$$(\lambda\delta_m\leqq|x|\leqq\pi,\ \lambda>1).$$

3. **Theorem 1.** *Let* $0<\alpha<1$ *and* $0<\beta<\min(1-\alpha,\ (2-\alpha)/3)$. *If*

(1) $$k^{2/(2-\alpha-3\beta)}<n_k<e^{2k/(2+\alpha+\beta)}$$

(2) $$|n_{k\pm1}-n_k|>4ekn_k^\beta$$

and

(3) $$\frac{1}{h^\beta}\int_0^h|f(t)-f(t\pm h)|\,dt=O(h^\alpha),$$

(4) $$\frac{1}{\tau}\int_0^\tau|f(t)-f(t\pm h)|\,dt=O(1),\ \textit{unif. in}\ \tau\geqq h^\beta,$$

then

(5) $$a_{n_k}=O(n_k^{-\alpha}),\qquad b_{n_k}=O(n_k^{-\alpha}),$$

where a_{n_k}, b_{n_k} *are non-vanishing Fourier coefficients of* $f(t)$.

Proof. Let $\delta_k=1/n_k^\beta$ and choose a sequence $M_k=[4ek/\delta_k]$. Let $T_{M_k}(x)$ be the trigonometrical polynomial determined by Lemma 1. Then we write, by (2),

$$\begin{aligned}a_{n_k}&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)T_{M_k}(t)\cos n_kt\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)-f(t+\pi/n_k)]T_{M_k}(t)\cos n_kt\,dt\\&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t+\pi/n_k)[T_{M_k}(t)-T_{M_k}(t+\pi/n_k)]\cos n_kt\,dt\\&=I_1+I_2\end{aligned}$$

and

$$\begin{aligned}I_1&=\frac{1}{2\pi}\Big[\int_{|t|\leqq\delta_k}+\int_{|t|>\delta_k}\Big]\Big[f(t)-f(t+\pi/n_k)\Big]T_{M_k}(t)\cos n_kt\,dt\\&=I_{11}+I_{12}.\end{aligned}$$

We have then

$$|I_{11}|\leqq\frac{A}{\delta_k}\int_{-\delta_k}^{\delta_k}|f(t)-f(t+\pi/n_k)|\,dt\leqq\frac{A}{n_k^\alpha}$$

by the condition (3) and Lemma 1, (i), and

$$|I_{12}|\leqq AM_k/\delta_ke^k\leqq A/n_k^\alpha$$

by (1) and Lemma 1, (ii). Further we write

$$\begin{aligned}I_2&=A\int_{-\pi}^{\pi}f(t+\pi/n_k)\Big[T_{M_k}(t)-T_{M_k}(t+\pi/n_k)\Big]\cos n_kt\,dt\\&=\frac{A}{2}\int_{-\pi}^{\pi}\Big[f(t+\pi/n_k)-f(t)\Big]\Big[T_{M_k}(t)-T_{M_k}(t+\pi/n_k)\Big]\cos n_kt\,dt\end{aligned}$$

$$+\frac{A}{2}\int_{-\pi}^{\pi} f(t)\Big[T_{M_k}(t-\pi/n_k)-2T_{M_k}(t)+T_{M_k}(t+\pi/n_k)\Big]\cos n_k t\,dt$$
$$=I_{21}+I_{22}.$$

Dividing the integral I_{21} into three parts, we get for a $\theta(0<\theta<1)$

$$|I_{21}|\leqq\frac{A}{n_k}\int_{-\pi}^{\pi}|f(t)-f(t+\pi/n_k)|\,|T'_{M_k}(t+\theta\pi/n_k)|\,dt$$
$$=\frac{A}{n_k}\Big[\int_{-\pi}^{-\lambda\delta_k}+\int_{-\lambda\delta_k}^{\lambda\delta_k}+\int_{\lambda\delta_k}^{\pi}\Big]|f(t)-f(t+\pi/n_k)|\,|T'_{M_k}(t+\theta\pi/n_k)|\,dt$$
$$=I_{211}+I_{212}+I_{213},$$

where $\lambda>1$ and

$$|I_{212}|\leqq A\frac{M_k}{n_k}\frac{1}{\delta_k}\int_{-\lambda\delta_k}^{\lambda\delta_k}|f(t)-f(t+\pi/n_k)|\,dt\leqq\frac{AM_k}{n_k^{1+\alpha}}\leqq\frac{A}{n_k^{\alpha}}$$

by (3) and Lemma 1, (iii), and putting $F(t)=\int_0^t|f(u)-f(u+\pi/n_k)|\,du$

$$|I_{213}|\leqq\frac{AM_k^2}{n_k\delta_k e^k}+\frac{A}{n_k}\int_{\lambda\delta_k}^{\pi}\frac{|f(t)-f(t+\pi/n_k)|}{t^2}\,dt$$
$$=\frac{AM_k^2}{n_k\delta_k e^k}+\frac{A}{n_k\delta_k^2}\int_0^{\lambda\delta_k}|f(t)-f(t+\pi/n_k)|\,dt+O\Big(\frac{1}{n_k}\Big)+\frac{A}{n_k}\int_{\lambda\delta_k}^{\pi}\frac{F(t)}{t^3}\,dt$$
$$<\frac{AM_k}{\delta_k e^k}+\frac{A}{n_k\delta_k}\leqq\frac{A}{n_k^{\alpha}}$$

by (1), (3), (4), and Lemma 1, (iv). I_{211} may also be estimated similarly to I_{213}. Thus we have

$$|I_{21}|\leqq A/n_k^{\alpha}.$$

Further we get

$$|I_{22}|\leqq AM_k^2/n_k^2\delta_k\leqq A/n_k^{\alpha}$$

by Lemma 1, (v).

Collecting above estimations we get the conclusion.

Theorem 2. *Let* $0<\alpha<1$, $0<\beta<(2-\alpha)/3$, *and*

$$\gamma>2/\min\,(1-\beta,\ 2-\alpha-3\beta)$$

(*or especially* $0<\beta<(1-\alpha)/2$ *and* $\gamma>2/(1-\beta)$). *If the Fourier coefficients of* $f(t)$ *vanish except for* $n=[k^{\gamma}]$ $(k=1,2,3,\ldots)$ *and the conditions* (3) *and* (4) *of Theorem 1 are satisfied, then* (5) *holds true.*

Proof runs similarly to that of Theorem 1, making use of Lemma 2 instead of Lemma 1. In this case

$$n_k=[k^{\gamma}],\ \delta_k=1/k^{\gamma\beta},\ M_k=4(ek)^{1+\gamma\beta}.$$

References

[1] M. E. Noble: Coefficient properties of Fourier series with a gap condition, Math. Ann., **128**, 55-62 (1954).

[2] A. Zygmund: Trigonometrical series, Warszawa (1935).