

16. On Newman Algebra. III

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Introduction

There are many kinds of postulates for Boolean algebra and Boolean ring as shown by D. G. Miller¹⁾ in his recent paper. The same holds also for Newman algebra which we considered in previous papers I, II.²⁾ (We remark that we call Newman algebra the algebraic system which is the direct union of a Boolean algebra (=Boolean lattice) and a Boolean ring (with unity), the latter satisfying also the associative law for multiplication, whereas usually the validity of this last law is not assumed in the definition of Newman algebra.) We have given two postulate-sets I*, II* for our Newman algebra respectively in I, II; now we propose to give another one III*.

In §1 below we shall give the complete list of Set III*, and prove the equivalence of Set III* with Set I*, wherewith the equivalence of all Sets I*, II*, III* will be established as we have already proved II* as equivalent with I*. In §2 we give the independence proofs. The eight-element system we use has been constructed by the same method as we have explained in another paper.³⁾

We shall give here the difference of our Set III* from Set I* and Set II*. We introduce a new postulate

$$6'. \quad a(b+c)=ca+ba$$

which will replace 3. $a+b=b+a$ and 6. $a(b+c)=ab+ac$ of Set I*, and 6. $a(b+c)=ab+ac$ and 8'. $a+b'b=b'b+a$ of Set II*. The form of postulate 6' was suggested by the postulate $a[(b+c)+d]=a(d+c)+ab$ of Miller.

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1. Postulates of Set III*

Our postulates are the propositions below on a class K , two binary operation $+$, \times and a unary operation \prime (in the postulates that are not existence postulates supply the condition: *if the elements indicated are in K*). It is to be remarked that the unary operation \prime is not required to be single-valued in our postulates.

Set III*

1. K is not empty.
2. If $a, b \in K$, then an element $a + b \in K$ is uniquely determined.
4. If $a, b \in K$, then an element $a \times b \in K$ is uniquely determined. (For the sake of brevity we shall write ab for $a \times b$.)
5. $a(bc) = b(ca)$.
- 6'. $a(b + c) = ca + ba$.
7. To each $a \in K$ corresponds at least one $a' \in K$.
8. $a + b'b = a$.
9. $a(b' + b) = a$.

We can easily obtain Set III* from Set I* as we have proved in I the commutativity of multiplication $ab = ba$ holds in Set I*.

Conversely, one can deduce Set I* from Set III*. Proof is very simple and can be formulated by the following theorems. Numbers in each proof of the theorems refer to postulates of Set III* but the use of postulates 1, 2, 4, 7 will be implicit. Theorems will be indicated by T .

$T1.$ $aa = a$.

Proof. $a = a(a' + a) = aa + a'a = aa$ by 9, 6', 8.

$T2.$ $ab = ba$.

Proof. $ab = (ab)(ab) = a[b(ab)] = a[a(bb)] = a(ab) = a(ba)$
 $= b(aa) = ba$ by $T1, 5, 5, T1, 5, 5, T1$.⁴⁾

$T3.$ $a + b = b + a$.

Proof. $a + b = (a + b)(a' + a) = (a' + a)(a + b)$
 $= b(a' + a) + a(a' + a) = b + a$ by 9, $T2, 6', 9$.

$T4.$ $a(b + c) = ab + ac$.

Proof. $a(b + c) = ca + ba = ac + ab = ab + ac$ by 6', $T2, T3$.

Now, as $T3$ and $T4$ are same as Postulates 3 and 6 of Set I* respectively, Sets I* and III* are equivalent.

If we introduce as in I, the postulates

$$10_1. \quad a + a = a,$$

$$10'_2. \quad (a' + a) + a = a',$$

then we obtain the following theorems.

$T5.$ The following set of postulates on K characterizes the Boolean lattice:

Set V: 1, 2, 4, 5, 6', 7, 8, 9, 10₁.

$T6.$ The following set of postulates on K characterizes the Boolean ring with unity:

Set VI: 1, 2, 4, 5, 6', 7, 8, 9, 10'₂.

2. Independence Proofs for Sets III*, V, VI

The independence of postulates of Set III* will follow from that of Sets V, VI, as these sets include our original postulate-set, Set III*.

The independence of postulates of Set V and Set VI will be established by the following examples: we shall list only the four-, eight-, and sixteen-element systems for the postulate 5 of Set V and the postulates 5 and 8 of Set VI. The independence of the postulate 1 in each Set V and Set VI is shown by the empty set K . For the remaining postulates in each set the examples are easy to give as two-element systems and they are to be omitted. We shall denote by $K_i\alpha$ an independence system for postulate α of K and for $i=V, VI, \alpha=1, 2, 4, 5, 6', 7, 8, 9, 10, 10'$; for example $K_{VI}5$ is an independence system of postulate 5 in K of Set VI.

$K_{VI}5:$	$+ \begin{array}{c c} 0 & 1 a b c \alpha \beta \gamma \\ \hline 0 & 0 1 a b c \alpha \beta \gamma \\ 1 & 1 0 a \beta \gamma a b c \\ a & a \alpha 0 \gamma \beta 1 c b \\ b & b \beta \gamma 0 \alpha c 1 a \\ c & c \gamma \beta \alpha 0 b a 1 \\ \alpha & \alpha a 1 c b 0 \gamma \beta \\ \beta & \beta b c 1 a \gamma 0 \alpha \\ \gamma & \gamma c b a 1 \beta \alpha 0 \end{array}$	$\times \begin{array}{c c} 0 & 1 a b c \alpha \beta \gamma \\ \hline 0 & 0 0 0 0 0 0 0 \\ 1 & 0 1 a b c \alpha \beta \gamma \\ a & 0 a a 1 1 0 a \alpha \\ b & 0 b 1 b 1 \beta 0 \beta \\ c & 0 c 1 1 c \gamma \gamma 0 \\ \alpha & 0 \alpha 0 \beta \gamma \alpha \gamma \beta \\ \beta & 0 \beta \alpha 0 \gamma \gamma \beta \alpha \\ \gamma & 0 \gamma \alpha \beta 0 \beta \alpha \gamma \end{array}$	$\begin{array}{c c} a & a' \\ \hline 0 & 1 \\ 1 & 0 \\ a & \alpha \\ b & \beta \\ c & \gamma \\ \alpha & a \\ \beta & b \\ \gamma & c \end{array}$
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Here $\alpha = \alpha(\beta\gamma) \neq \beta(\gamma\alpha) = \beta$.

$K_{VI}8:$	$+ \begin{array}{c c} 0 & a b c \\ \hline 0 & 0 a b c \\ a & a 0 c b \\ b & b c 0 a \\ c & c b a 0 \end{array}$	$\times \begin{array}{c c} 0 & a b c \\ \hline 0 & 0 0 0 0 \\ a & 0 c b a \\ b & 0 b 0 b \\ c & 0 a b c \end{array}$	$\begin{array}{c c} a & a' \\ \hline 0 & c \\ a & b \\ b & a \\ c & 0 \end{array}$
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Here $b = 0 + ab \neq 0$.

K_V5 : This system is composed of sixteen elements and is constructed as follows:

Let \mathfrak{A} be the two-element Boolean lattice $\{0, 1\}$ with the well-known operations \vee, \wedge, \prime , and \mathfrak{B} the eight-element system used as $K_{VI}5$ above. Let \mathfrak{S} be the direct product $\mathfrak{A} \times \mathfrak{B}$ of these sets $\mathfrak{A}, \mathfrak{B}$, so that $\mathfrak{S} \ni (\xi, \eta), \xi \in \mathfrak{A}, \eta \in \mathfrak{B}$, and $(\xi_1, \eta_1) = (\xi_2, \eta_2)$ if and only if $\xi_1 = \xi_2, \eta_1 = \eta_2$. The operations $+, \times, \prime$ will be introduced into \mathfrak{S} by the rules:

$$\begin{aligned}
 (\xi, \eta) + (\xi, \eta) &= (\xi, \eta), \\
 (\xi_1, \eta_1) + (\xi_2, \eta_2) &= (\xi_1 \vee \xi_2, \eta_1 + \eta_2) && \text{if } \eta_1 \neq \eta_2, \\
 (\xi_1, \eta_1) \times (\xi_2, \eta_2) &= (\xi_1 \wedge \xi_2, \eta_1 \times \eta_2), \\
 (\xi, \eta)' &= (\xi', \eta').
 \end{aligned}$$

Then K_V5 is given by \mathfrak{S} with these operations.⁵⁾

References

- 1) D. G. Miller: Postulates for Boolean algebra, Amer. Math. Monthly, **59**, 93-96 (1952).

- 2) Y. Wooyenaka: On Newman algebra, Proc. Japan Acad., **30**, 170-175 (1954). On Newman algebra. II, Proc. Japan Acad., **30**, 562-565 (1954).
 3) Y. Wooyenaka: On Postulate-sets for Newman algebra and Boolean algebra: to be forthcoming.

- 4) Another proof for $T2$, $ab=ba$.

We prove the following theorem first.

$T0$. $a'a=aa'$.

Proof. $a'a=(a'a)(a'+a)=a(a'a)+a'(a'a)=a'(aa)+a'(aa')=a(aa')+a'(aa')$
 $=(aa')(a'+a)=aa'$ by 9, 6', 5, 5, 6', 9.

Now the proof of $T2$ is given as follows:

$ab=(ab)(b'+b)=b(ab)+b'(ab)=a(bb)+a(b'b)=b(ba)+b'(ba)=(ba)(b'+b)=ba$
 by 9, 6', 5- $T0$, 5, 6', 9.

- 5) We note that in $K_{V1}5$ the associative law for addition holds, but not the associative law for multiplication, while both laws do not hold in K_V5 .