37. *On Newman Algebra*

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Introduction

As is well known, Newman has given an elegant system of postulates for an algebraic system which is the direct union of the Boolean subalgebra of elements satisfying $a+a=a$, and the Boolean subring satisfying $a+a=0$.[1] But the Boolean subring which appears here satisfies all postulates for the Boolean ring with unity given by Stone[2] except the associative law for multiplication; this ring is the so-called "non associative Boolean ring". In fact, an example[3] can be given of Newman algebra with eight elements inclusive of 0 and 1, whose every element satisfies $a+a=0$, and which is really a non associative Boolean ring with unity.

It would be of interest to consider an algebraic system, analogous to Newman's which is the direct union of a Boolean algebra (= Boolean lattice) and a Boolean ring (with unity), the latter satisfying also the associative law for multiplication. In this paper, such an algebraic system is called *Newman algebra* and it will be characterized by an independent set of postulates.

We first show in § 1 the postulates of our new algebraic system in which the existence of special elements 0 and 1 is not postulated, and the cyclic associative law for multiplication is adopted. Moreover, we shall show by a very simple proof that each subalgebra of even or odd elements in the direct decomposition theorem is a Boolean lattice or a Boolean ring with unity respectively. As a byproduct we shall obtain new postulate-sets for the Boolean lattice and the Boolean ring. In § 2 we shall give the independence proofs for these new postulate-sets, as well as the postulate-set for our Newman algebra.

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1. The Postulates and Elementary Properties

Our postulates are the propositions below on a *class K*, a *binary operation* $+$, a *binary operation* $\times$, and a *unary operation* $'$ (in the postulates that are not existence postulates supply the condition: *if the elements indicated are in K*). It is to be remarked that the unary operation $'$ is not required to be single-valued in our postulates.

System $(K, +, \times, ')$

1. K is not empty.
2. If $a, b \in K$ then an element $a+b \in K$ is uniquely determined.
3. $a+b=b+a$.
4. If $a, b \in K$ then an element $a \times b \in K$ is uniquely determined.

(For the sake of brevity we shall write ab for $a \times b$.)

5. $a(bc)=b(ca)$.
6. $a(b+c)=ab+ac$.
7. To each $a \in K$ corresponds at least one $a' \in K$.
8. $a+b'b=a$.
9. $a(b'+b)=a$.

We shall now derive from these postulates several elementary properties of the system K. In each proof of the theorems, we shall list the numbers of axioms used in the transformation of formulas, but the use of the postulates 1, 2, 4, 7 will be implicit in general. Theorems will be indicated by T, lemmas by L, definitions by D, hypothesis by H in the following.

$T0$. $a'a=aa'$.

Proof. $a'a=a'(a+a'a)=a'a+a'(a'a)=a'(a'a)=a'(aa')=a(a'a')$ $=a(a'a'+a'a)=a\{a'(a'+a)\}=aa'$ by 8, 6, 3−8, 5, 5, 8, 6, 9.

$T1$. $aa=a$.

Proof. $a=a(a'+a)=aa'+aa=aa+a'a=aa$ by 9, 6, 3−$T0$, 8.

$T2$. $ab=ba$.

Proof. $ab=(ab)(ab)=a(b(ab))=a(a(bb))=a(ab)=a(ba)=b(aa)=ba$ by $T1$, 5, 5, $T1$, 5, 5, $T1$.

$T3$. a' is unique.

Proof. Let a_1' and a_2' be two elements corresponding to a by 7, then

$$a_1'=a_1'(a_2'+a)=a_1'a_2'+a_1'a=a_1'a_2'=a_2'a_1'+a_2'a=a_2'(a_1'+a)=a_2'$$

by 9−H, 6, 8, $T2$−8−H, 6, 9.

$T4$. $aa'=a'a=b'b$ for any $a, b \in K$, thus the element $a'a=aa'$ is independent of a.

Proof. $aa'=a'a=a'a+b'b=b'b+a'a=b'b$ by $T0$, 8, 3, 8.

$D1$. The element $a'a=aa'$ is denoted by 0.

$T5$. $a+a'=a'+a=b'+b$ for any a, $b \in K$, thus the element $a'+a=a+a'$ is independent of a.

Proof. $a+a'=a'+a=(a'+a)(b'+b)=(b'+b)(a'+a)=b'+b$ by 3, 9, $T2$, 9.

$D2$. The element $a'+a=a+a'$ is denoted by 1.

$T6$. $a+0=0+a=a$.

Proof. This follows from $8-D1-3$.

$T7.$ $a1=(1a=)\,a.$

Proof. This follows from $9-D2(-T2)$.

$T8.$ $(a+b)c=ac+bc.$

Proof. $(a+b)c=c(a+b)=ca+cb=ac+bc$ by $T2$, 6, $T2$.

$T9.$ To each a corresponds at least one a', such that $aa'=0$ and $a+a'=1$.

Proof. It follows from 7, $4-D1$ and $2-D2$.

(In fact a' is unique by $T3$.)

$T10.$ $a(bc)=(ab)c.$

Proof. $a(bc)=b(ca)=c(ab)=(ab)c$ by 5, 5, $T2$.

$L1.$ $(a+b)+c=(a'b+a'c)+a\{(1+b)+c\}.$

Proof. $(a+b)+c=(a'+a)\{(a+b)+c\}=a'\{(a+b)+c\}+a\{(a+b)+c\}$
$=\{(a'a+a'b)+a'c\}+\{(aa+ab)+ac\}=(a'b+a'c)+\{(a1+ab)+ac\}$
$=(a'b+a'c)+a\{(1+b)+c\}$
by $D2-T7$, $T8$, $6-6$, $D1-T6-T1-T7$, $6-6$.

$L2.$ $(a+b)+c=(c'a+c'b)+\{(ca+cb)+c\}.$

Proof. $(a+b)+c=(c'+c)\{(a+b)+c\}$
$=c'\{(a+b)+c\}+c\{(a+b)+c\}=\{(c'a+c'b)+c'c\}+\{(ca+cb)+cc\}$
$=(c'a+c'b)+\{(ca+cb)+c\}$ by $D2-T7$, $T8$, $6-6$, $8-T1$.

$T11.$ $(a+b)+c=a+(b+c).$

Proof.
$$
\begin{aligned}
(a+b)+c&=(a'b+a'c)+a\{(b+1)+c\}\\
&=(a'b+a'c)+a[(b'1+b'c)+b\{(1+1)+c\}]\\
&=(a'b+a'c)+a\Big[(b'1+b'c)+b[(c'1+c'1)+\{(c1+c1)+c\}]\Big]\\
&=(a'b+a'c)+a\Big[(b'1+b'c)+b[(c'1+c'1)+c\{(1+1)+1\}]\Big]\\
&=(a'b+a'c)+a[(b'1+b'c)+b\{(c+1)+1\}]\\
&=(a'b+a'c)+a[(b'c+b'1)+b\{(1+c)+1\}]\\
&=(a'b+a'c)+a\{(b+c)+1\}\\
&=(a'b+a'c)+\{(ab+ac)+a\}=(b+c)+a\\
&=a+(b+c)
\end{aligned}
$$
by $L1-3$, $L1$, $L2$, $T7-6-6$, $L1$, 3, $L1$, $6-6-T7$, $L2$, 3.

Now K is a Newman algebra according to Birkhoff's definition;[1] this follows from postulate 6 and Theorems 8, 7, 6, 9. And moreover, our Newman algebra is commutative and associative for multiplications by $T2$ and $T9$ respectively. Using Birkhoff's argument[1] it is easy to see that our system K is the direct union of the subalgebras of even and odd elements. We shall not repeat here the whole part of the proof,[1] only we shall give in the following a simple proof of the fact that each subalgebra of even or odd elements, K_1 or K_2, forms a Boolean lattice or a Boolean ring with unity[2] respectively.

$T12$. The system K_1 which satisfies the postulates $1-9$ and

$$10_1. \qquad a+a=a$$

is a Boolean lattice.

Proof. (1) $1+a=a+1=a+(a+a')=(a+a)+a'=a+a'=1$ by 3, $D2$, $T11$, 10_1, $D2$.

(2) $a+ab=a1+ab=a(1+b)=a1=a$ by $T7$, 6, (1), $T7$.

(3) $a(a+b)=aa+ab=a+ab=a$ by 6, $T1$, (2).

(4) $(a+b)(a+c)=a(a+c)+b(a+c)$
$=a+(ba+bc)=(a+ab)+bc$
$=a+bc$ by $T8$, (3)$-$6, $T11-T2$, (3).

Here we have the standard postulates for Boolean lattice: 10_1, $T1$, 3, $T2$, $T11$, $T10$, (2), (3), 6, (4), and $T9$, therefore K_1 is a Boolean lattice.

Thus we have proved

$T13$. The following set of postulates on K characterizes the Boolean lattice:

Set I: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10_1.

$T14$. The system K_2 which satisfies the postulates $1-9$ and

$$10_2. \qquad a+a=0$$

is a Boolean ring with unity.

Proof. (5) For every $a,\ b\in K_2$, the equation $x+a=b$ has a solution in K_2.

$\because$ If $a,\ b\in K$ then $b+a\in K$ by 2. Put $x=b+a$ then
$x+a=(b+a)+a=b+(a+a)=b+0=b$ by H, $T11$, 10_2, $T6$.

Therefore $b+a$ is a solution (and the solution is unique by 2).

Now, 3, $T11$, (5), $T10$, 6, $T8$, $T1$, $T7$ are the postulates for Boolean ring with unity due to Stone [2): p. 39]. Therefore K_2 is a Boolean ring with unity.

Here we see that our postulate-set is sufficient to characterize the Newman algebra in the sense described in Introduction.

For the further discussion we give attention to the following theorem.

$T15$. If $a+a=0$ holds for all a of a Newman algebra according to Birkhoff's definition then

$$10_2'. \qquad (a'+a)+a=a'$$

holds for all a as well as for all a' for this Newman algebra. The converse also holds.

Proof. $[(a'+a)+a=\{(a'+a)+a\}1=\{(a'+a)+a\}(a+a')$
$=\{(a'+a)+a\}a+\{(a'+a)+a\}a'=\{(a'a+aa)+aa\}+\{(a'a'+aa')+aa'\}$
$=\{(0+a)+a\}+\{(a'+0)+0\}]=(a+a)+a'=0+a'=a'$

by $N2$, $N4$, $N1$, $N1'-N1'$, $(T1)-(N4')-N4$, $N3$, H, $N3$ [1): pp. 155-156].

Conversely, $a+a=1a+aa=(a'+a)a+aa=\{(a'+a)+a\}a=a'a=0$

by $(N2')-(T1)$, $(N4')$, $N1'$, H, $(N4')$.

Thus

$T16$. The following set of postulates on K characterizes the Boolean ring with unity:

Set II: 1, 2, 3, 4, 5, 6, 7, 8, 9, $10_2'$.

2. The Independence Proofs

As the Sets I, II include our original postulate-set $\{1, 2, \ldots, 9\}$ the independence of our postulates 1, 2, ..., 9 will follow from that of Sets I, II.

The independence of the postulates of Set I and Set II will be established by the following examples; we shall list only the four-, and eight-element systems for the postulate 5 of Set I and the postulates 5 and 8 of Set II. The independence of the postulate 1 in each Set I and Set II is shown by the empty set K. For the remaining postulates in each set the examples are easy to give as two-element systems and they are to be omitted. We shall denote by $K_i\alpha$ an independence system for postulate α of K and for $i=I, II$, $\alpha=1, 2, \ldots, 9, 10_1, 10_2$; for example K_I5 is an independence system of postulate 5 in K of Set I.

K_I5:

+	0	1	a	b	c	α	β	γ
0	0	1	a	b	c	α	β	γ
1	1	1	1	1	1	1	1	1
a	a	1	a	c	c	1	β	β
b	b	1	c	b	c	α	1	α
c	c	1	c	c	c	1	1	1
α	α	1	1	α	1	α	1	α
β	β	1	β	1	1	1	β	β
γ	γ	1	β	α	1	α	β	γ

×	0	1	a	b	c	α	β	γ
0	0	0	0	0	0	0	0	0
1	0	1	a	b	c	α	β	γ
a	0	a	a	0	a	0	a	0
b	0	b	0	b	b	b	0	0
c	0	c	0	c	c	c	0	0
α	0	α	0	b	b	α	γ	γ
β	0	β	a	0	a	γ	β	γ
γ	0	γ	0	0	0	γ	γ	γ

a	a'
0	1
1	0
a	α
b	β
c	γ
α	a
β	b
γ	c

$K_{II}5$:

+	0	1	a	b	c	α	β	γ
0	0	1	a	b	c	α	β	γ
1	1	0	α	β	γ	a	b	c
a	a	α	0	c	b	1	γ	β
b	b	β	c	0	a	γ	1	α
c	c	γ	b	a	0	β	α	1
α	α	a	1	γ	β	0	c	b
β	β	b	γ	1	α	c	0	a
γ	γ	c	β	α	1	b	a	0

×	0	1	a	b	c	α	β	γ
0	0	0	0	0	0	0	0	0
1	0	1	a	b	c	α	β	γ
a	0	a	a	0	a	0	a	0
b	0	b	0	b	b	b	0	0
c	0	c	0	c	c	c	0	0
α	0	α	0	b	b	α	γ	γ
β	0	β	a	0	a	γ	β	γ
γ	0	γ	a	a	0	γ	β	γ

a	a'
0	1
1	0
a	α
b	β
c	γ
α	a
β	b
γ	c

Here $0=c(a\beta) \neq a(\beta c)=a$ both in K_I5 and $K_{II}5$.

$K_{II}8$:

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

×	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	1	b
b	0	b	b	0

a	a'
0	1
1	0
a	b
b	a

Here $b=0+ab \neq 0$.

References

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