90. On Cauchy's Problem in the Large for Wave Equations.

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§1. *Introduction.* Let R be a connected domain of an orientable, m-dimensional Riemannian space with the metric $ds^2=g_{ij}(x)dx^i dx^j$. We consider the wave equation

$$(1.1)\qquad \frac{\partial^2 u(x,t)}{\partial t^2}=A_x u(x,t), -\infty<t<\infty,$$

with Cauchy's data

$$(1.2)\qquad u(x,0)=f(x),\quad \frac{\partial u(x,0)}{\partial t}=h(x).$$

Here the differential operator $A=A_x$ defined by

$$(1.3)\qquad A_x f(x)=b^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j}+a^i(x)\frac{\partial f(x)}{\partial x^i}+e(x)f(x)$$

is *elliptic* in the sense that the quadratic form $b^{ij}(x)\xi_i\xi_j$ is >0 for $\sum_i(\xi_i)^2>0$. Since the value of $A_x f(x)$ must be independent of the local coordinates $(x^1, \dots, x^m)$ of the point x, the coefficients $a^i(x)$ and $b^{ij}(x)$ must be transformed, by the coordinates change $x\to\bar{x}$, respectively into

$$(1.4)\qquad \bar{a}^i(\bar{x})=\frac{\partial \bar{x}^i}{\partial x^k}a^k(x)+\frac{\partial^2\bar{x}^i}{\partial x^k\partial x^s}b^{ks}(x) \text{ and } \bar{b}^{ij}(\bar{x})=\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^s}b^{ks}(x).$$

For the sake of simplicity, we assume that $g_{ij}(x)$, $b^{ij}(x)$, $a^i(x)$ and $e(x)$ are infinitely differentiable functions of the local coordinates $(x^1, \dots, x^m)$.

Since we are concerned with *the existence in the large of the integral* of (1.1)–(1.2), it will perhaps be necessary to rely upon operator-theoretical method[1]. We here assume that the operator A_x is, as in the case of Laplacian, *formally self-adjoint* and *non-positive definite*, viz.

$$(1.5)\qquad \int_R (A_x f(x))h(x)dx=\int_R f(x)(A_x h(x))dx \text{ and } \int_R (A_x f(x))f(x)dx\leqq 0$$
$$(dx=\sqrt{g(x)}\,dx^1\dots dx^m,\quad g(x)=det(g_{ij}(x)),$$

if $f(x)$ and $h(x)$ are twice continuously differentiable such that $f(x)$ vanishes outside a compact set contained in the interior of R. Then we may integrate, by virtue of the Hilbert space technique, an operator-theoretical variant of (1.1)–(1.2) It will next be shown, by a parametrix consideration, that this operator-theoretical integral is, for sufficiently smooth initial data (1.2), equivalent to the ordinary integral of the genuine differential equation (1.1)–(1.2). It is to be noted that the Lemma 2 below, which is of the type of Poisson's equation, may be of use in other problems relating to the elliptic differential operator.

§ 2. *An operator-theoretical integration.* Let L be the linear space of twice continuously differentiable real-valued functions $f(x)$ vanishing outside compact set and satisfying a certain linear boundary condition on the boundary ∂R of R. It is assumed that the boundary condition is chosen in such a way that we have

$$\int_R (A_x f(x))h(x)dx = \int_R f(x)(A_x h(x))dx \text{ and} \tag{2.1}$$

$$\int_R (A_x f(x))f(x)dx \leqq 0 \quad \text{for } f,\ h \in L. \tag{2.2}$$

Such boundary condition is possible because of the assumption (1.5). L is a pre-Hilbert space by the norm

$$\|f\| = \left(\int_R f(x)^2 dx\right)^{1/2} = (f, f)^{1/2}, \tag{2.3}$$

such that the completion L^a of this linear normed space L is a real Hilbert space $L_2(R)$.

We consider $A = A_x$ to be an additive operator defined on $L \subseteqq L^a$ into L^a. Let $\tilde{A}$ be a non-positive definite self-adjoint extension of A. Such $\tilde{A}$ may be defined as follows[2]: Let L' be the completion of the linear space L by the new metric

$$\|f\|' = ((-Af, f) + (f, f))^{1/2}. \tag{2.4}$$

Because of (2.2), we may identify L' with a linear subspace of L^a. Then

(2.5) $\tilde{A}$ is the contraction of the adjoint operator A^* of A restricted to the domain $D(\tilde{A}) = L' \frown D(A^*)$, where $D(A^*)$ is the domain of A^*.

We have, by (2.1),

$$L \subseteqq D(\tilde{A}). \tag{2.6}$$

Let (2.7):
$$-\tilde{A} = \int_0^\infty \lambda dE(\lambda)$$

be the spectral resolution of $-\tilde{A}$ and let

$$(-\tilde{A})^{1/2} = \int_0^\infty \lambda^{1/2} dE(\lambda) \tag{2.8}$$

be the positive square root of the operator $-\tilde{A}$. Surely we have

(2.9) the domain $D((-\tilde{A})^{1/2})$ of $(-\tilde{A})^{1/2} \supseteqq D(\tilde{A})$, and hence, by (2.6),

$$L \subseteqq D(\tilde{A}) \subseteqq D((\tilde{A})^{1/2}). \tag{2.6$'$}$$

Let us consider, for f and $h \in L$,

$$\begin{aligned} \tilde{u}(x, t) &= (\cos(-\tilde{A})^{1/2}t)f(x) + (\sin((-\tilde{A})^{1/2}t)/(-\tilde{A})^{1/2})h(x) \\ &= \int_0^\infty \cos(\lambda^{1/2}t)dE(\lambda)f(x) + \int_0^\infty (\sin(\lambda^{1/2}t)/\lambda^{1/2})dE(\lambda)h(x). \end{aligned} \tag{2.10}$$

The convergence of the right hand integral is clear. We see, by (2.6)$'$, that $\tilde{u}(x, t)$ satisfies the operator-theoretical differential equation

$$(2.11)\qquad \partial_t\partial_t u(x, t)=\tilde{A}_x\tilde{u}(x, t),\ \text{where}\quad \partial_t\tilde{u}(x, t)=\text{strong}\lim_{\delta\to 0}\delta^{-1}(\tilde{u}(x, t+\delta)-\tilde{u}(x, t)).$$

We have also (2.12): $\tilde{u}(x, 0)=f(x),\quad \partial_t\tilde{u}(x, 0)=h(x).$
Therefore we have:

Theorem 1. (2.10) is an operator-theoretical solution of Stokes' type of the operator-theoretical variant (2.11)–(2.12) of (1.1)–(1.2).

Let D be the subset of L consisting of all the infinitely differentiable functions $f(x)$ such that $f(x)\in$ the domain $D(\tilde{A}_x^q)$ of the operator $\tilde{A}_x^q$ for every $q>0$. Such is the case for infinitely differentiable function $f(x)$ when $f(x)$ vanishes outside a compact set contained in the interior of R. From the definition (2.10) and (2.6)$'$, we see that (2.13): if f and h are both in D, the function $\tilde{u}(x, t)$ given by (2.10) is in the domain $D(\tilde{A}_x^q)$ for any $q>0$. We will show, in § 4, that such $\tilde{u}(x, t)$ is equal (x, t)-almost everywhere to a function $u(x, t)$ which is infinitely differentiable in (x, t), so that $u(x, t)$ is an ordinary integral of the genuine differential equation (1.1)–(1.2).

§ 3. *The parametrix for the iterated elliptic operator.* The hypothesis of the formal self-adjointness of the operator $A=A_x$ is not needed in this §. Thus let

$$(3.1)\qquad A'_x f(x)=b^{ij}(x)\frac{\partial^2 f}{\partial x^i\partial x^j}+c^i(x)\frac{\partial f}{\partial x^i}+p(x)f(x)$$

be the formally adjoint operator of A_x. We will construct a parametrix for the iterated elliptic operator (3.2): $A_x'^{q-1}$. To this purpose, let $\Gamma(P, Q)=r(P, Q)^2$ be the square of the smallest distance between the two points $P=(x'^1, \dots, x'^m)$ and $Q=(x^1, \dots, x^m)$ of R according to the new metric (3.3): $dr^2=b_{ij}(x)dx^i dx^j$, where $(b_{ij}(x))=(b^{ij}(x))^{-1}$.
We have then:

Lemma 1[3]. *Let the dimension m be odd. For any positive integer n and for any even $\alpha\geqq 0$, we may construct a parametrix $W_\alpha(P, Q)$ for the operator $A'=A'_x$:*

$$(3.4)\quad W_\alpha(P, Q)=\sum_{k=0}^{n}\Gamma(P, Q)^{(\alpha+2k-m)/2}V_k(P, Q)/K_m(\alpha)L_m(\alpha+2k),$$

where $K_m(\alpha)=2^{\alpha/2}\Gamma(\alpha/2),\ L_m(\alpha+2k)=2^{(\alpha+2k)/2}\Gamma((\alpha+2k+2-m)/2)$ *and $V_k(P, Q)$ are infinitely differentiable in the vicinity of $Q=P$ and* $V_0(P, P)=1$
so that (3.5):

$$A'_x W_{\alpha+2}(P, Q)=W_\alpha(P, Q)+\Gamma(P, Q)^{(\alpha+2+2n-m)/2}A'_x W_\alpha(P, Q)/K_m(\alpha+2)L_m(\alpha+2+2n)).$$

Proof. We introduce the normal coordinates y of $Q=(x^1, \dots, x^m)$ in the vicinity of P:

$$(3.6)\qquad y^\sigma=(\Gamma(P, Q))^{1/2}\left(\frac{dx^\sigma}{dr}\right)_{r=0}.$$

Let (3.7): $dr^2=\beta_{ij}(y)dy^i dy^j$.

We have the well-known formulae

(3.8) $\Gamma(P, Q)=\beta_{ij}(0)y^i y^j, \quad \beta_{ij}(y)y^j=\beta_{ij}(0)y^j$.

By virtue of (3.8), the operator

$$(3.9)\qquad A'=A'_y=\beta^{ij}(y)\frac{\partial^2}{\partial y^i\partial y^j}+\alpha^i(y)\frac{\partial}{\partial y^i}+\gamma(y)$$

$$((\beta^{ij}(y))=(\beta_{ij}(y))^{-1}),$$

when applied to the function of the form $f(\Gamma(P, Q), y)$, may be written as follows:

$$(3.10)\qquad A'_y f=4\Gamma\frac{\partial^2 f}{\partial \Gamma^2}+4y^\sigma\frac{\partial^2 f}{\partial\Gamma\partial y^\sigma}+M\frac{\partial f}{\partial\Gamma}+N(f), \quad \text{where}$$

$$M=\beta^{ij}\frac{\partial^2\Gamma}{\partial y^i\partial y^j}+\alpha^i\frac{\partial\Gamma}{\partial y^i}=2m+0(y), \quad N(f)=\beta^{ij}\frac{\partial^2 f}{\partial y^i\partial y^j}+\alpha^i\frac{\partial f}{\partial y^i}+\gamma f.$$

The differentiation in A'_y and in $N(f)$ are to be performed as if Γ and y^σ are independent variables. Hence, by

(3.11) $\alpha/K_m(\alpha+2)=1/K_m(\alpha), \quad (\alpha+2-m)/L_m(\alpha+2)=1/L_m(\alpha)$,

we obtain
$$A'_y W_{\alpha+2}(P, Q)=\sum_{k=0}^{n}\frac{\Gamma(P, Q)^{(\alpha+2k-m)/2}}{K_m(\alpha+2)L_m(\alpha+2k)}$$
$$\times\left\{2y^\sigma\frac{\partial V_k}{\partial y^\sigma}+\left(\frac{M}{2}+2k-m+\alpha\right)V_k+A'_y V_{k-1}(P,Q)\right\}$$
$$+\frac{\Gamma(P, Q)^{(\alpha+2+2n-m)/2}}{K_m(\alpha+2)L_m(\alpha+2+2n)}A'_y V_n$$
$$=W_\alpha(P, Q)+\frac{\Gamma(P, Q)^{(\alpha+2+2n-m)/2}}{K_m(\alpha+2)L_m(\alpha+2+2n)}A'_y V_n,$$

if $V_k(P, Q)$ may be so determined that $V_k(P, Q)$ are infinitely differentiable in the vicinity of $Q=P$, $V_{-1}(P, Q)\equiv 0$, $V_0(P, P)=1$ and

$$(3.12)\qquad 2y^\sigma\frac{\partial V_k}{\partial y^\sigma}+(\frac{M}{2}+2k-m)V_k(P, Q)+A'_y V_{k-1}(P, Q)$$
$$=0, \quad (k=0, 1, \ldots, n).$$

Such $V_k(P, Q)$ exist by virtue of the order relation

(3.13) $M=2m+0(y)$.

Proof. By putting $y^\sigma=r\eta^\sigma$, (3.12) is reduced to the ordinary differential equation in r containing the parameters η:

$$(3.12)'\quad 2r\frac{dV_k(P, r\eta)}{dr}+(\frac{M(r\eta)}{2}+2k-m)V_k(P, r\eta)=-A'_y V_{k-1}(P, r\eta).$$

Hence, by $V_{-1}(P, Q)\equiv 0$ and $V_0(P, P)=1$, we obtain

$$(3.14)\qquad V_0=\exp(-\int_0^r(2t)^{-1}(\frac{M}{2}-m)dt),$$
$$V_k=-V_0 r^{-k}\int_0^r t^{k-1}V_0^{-1}A'_y V_{k-1}dt.$$

Corollary.

$$(3.15)\qquad A'^{q-l}_y W_{2q}(P, Q)=W_{2l}(P, Q)+0(\Gamma(P, Q)^{(2l+2+2n-m)/2}),$$
$$A'^q_y W_{2q}(P, Q)=0(\Gamma(P, Q)^{(2+2n-m)/2}) \text{ for } P\neq Q.$$

Next let P_0 be any inner point of R and consider, for sufficiently small $\varepsilon>0$,

(3.16) $U_\alpha(P,Q)=W_\alpha(P,Q)\delta(\Gamma(P,Q))\delta(\Gamma(P_0,P))$, where $\delta(x)\geqq$ is infinitely differentiable in $x\geqq 0$ such that $\delta(x)=1$ or 0 according as $x\leqq\varepsilon$ or $x\geqq 2\varepsilon$.

Thus, in a certain vicinity of P_0,

(3.17) $$A'^{q-1}_y U_{2q}(P,Q)=U_{2i}(P,Q)+0(\Gamma(P,Q)^{(2i+2+2n-m)/2}),$$
$$A'^{q}_y U_{2q}(P,Q)=0(\Gamma(P,Q)^{(2+2n-m)/2}) \text{ for } P\neq Q.$$

After these preliminaries, we may prove an analogue of Poisson's equation, viz.

Lemma 2. Let the dimension m be odd and $\geqq 2$, and let $k(Q)$ be $\in L$. Then we have, for $2n\geqq m$,

(3.18) $C(P)k(P)=\int_R (A'^{q-1}_y U_{2q}(P,Q))(A_y k(Q))dQ$, *where $C(P)$ is infinitely differentiable and $\neq 0$ in a certain vicinity of P_0.*

Proof. We have, by Green's integral theorem and (3.17),

$$\int_R (A'^{q-1}_y U_{2q}(P,Q)(A_y k(Q))dQ$$
$$=\lim_{\kappa\to 0}\int_{R-\{Q;\Gamma(P,Q)\leqq\kappa\}} (A'^{q-1}_y U_{2q}(P,Q))(A_y k(Q))dQ$$
$$=\lim_{\kappa\to 0}\int_{R-\{Q;\Gamma(P,Q)\leqq\kappa\}} (A'_y(A'^{q-1}_y U_{2q}(P,Q))k(Q)dQ$$
$$+\lim_{\kappa\to 0}\int_{\Gamma(P,Q)=\kappa}\left\{\frac{A'^{q-1}_y U_{2q}(P,Q)}{\partial\nu}k(Q)-(A'^{q-1}_y U_{2q}(P,Q))\frac{\partial k(Q)}{\partial\nu}\right\}dS$$

where ν is the transversal direction defined by

(3.19) $$\frac{\partial\nu}{\partial y^i}=(\sqrt{g(y)}\,\beta^{ij}(y)\cos(r,y^j))^{-1},\quad (i=1,2,\dots,m)$$

and dS is the hypersurface element on $\Gamma(P,Q)=\kappa$.

We have, from (3.17),

$A'^{q}_y U_{2q}(P,Q)=0(\Gamma(P,Q)^{(2+2n-m)/2})$ for $P\neq Q$,

$A'^{q-1}_y U_{2q}(P,Q)=(4\Gamma((4-m)/2))^{-1}\Gamma(P,Q)^{(2-m)/2}+0(\Gamma(P,Q)^{(2+2n-m)/2})$.

Hence we have, when $\Gamma(P,Q)=\kappa$ tends to zero

$$\frac{\partial A'^{q-1}_y U_{2q}(P,Q)}{\partial\nu}\doteqdot(8\Gamma((4-m)/2)^{-1}(2-m)\Gamma^{-m/2}\frac{\partial\Gamma}{\partial y^i}\sqrt{g(y)}\,\beta^{ij}(y)\cos(r,y^j)$$
$$=(4\Gamma(4-m)/2)^{-1}(2-m)\Gamma^{-m/2}\beta_{ik}(0)y^k\sqrt{g(y)}\,\beta^{ij}(y)\cos(r,y^j)\quad\text{(by (3.8))}$$
$$=(4\Gamma((4-m)/2)^{-1}(2-m)y^j\Gamma^{-m/2}\sqrt{g(y)}\cos(r,y^j)\quad\text{(by (3.8))}$$
$$=(4\Gamma((4-m)/2)^{-1}(2-m)\Gamma^{(1-m)/2}\sqrt{g(r\eta)}\sum_{j=1}^m(\eta^j)^2\ \text{(by putting } y^j=r\eta^j).$$

Therefore we have $\int_R (A'^{q-1}_y U_{2q}(P,Q)(A_y k(Q))dQ$

$$=\lim_{\kappa\to 0}\int_{\beta_{ij}(P)\eta^i\eta^j=1}(4\Gamma((4-m)/2)^{-1}(2-m)\kappa^{(1-m)/2}\sqrt{g(\sqrt{\kappa}\,\eta)}\sum_{j=1}^m(\eta^j)^2 dS_{\sqrt{\kappa}\,\eta}$$
$$=(4\Gamma(4-m)/2)^{-1}(2-m)\sqrt{g(P)}\int_{\beta_{ij}(P)\eta^i\eta^j=1}\sum_{j=1}^m(\eta^j)^2 dS_\eta.$$

This proves (3.16).

§ 4. *The differentiability of the operator-theoretical solution* $\tilde{u}(Q,t)$. We first remark that we are dealing with the case $A'=A$. We will prepare two lemmas.

Lemma 3. For fixed t, there exists a sequence of functions $\{k_i(Q)\}\subseteqq L$ *such that*

(4.1) $$\text{strong}\lim_{i\to\infty} k_i(Q)=\tilde{u}(Q,t),$$

$$\lim_{i\to\infty}\int_R w(Q)(A_y k_i(Q))dQ=\int_R w(Q)(\tilde{A}_y\tilde{u}(Q,t))dQ \text{ for every } w(Q)\in L.$$

Proof. By $\tilde{u}(Q,t)\in D(\tilde{A}_y)$ and the definition (2.5) of $\tilde{A}$, there exists a sequence of functions $\{k_i(Q)\}\subseteqq L$ such that strong $\lim_{i\to\infty} k_i(Q)=\tilde{u}(Q,t)$. We have, for any $w(Q)\in L$,

$$\lim_{i\to\infty}\int_R w(Q)(A_y k_i(Q))dQ=\lim_{i\to\infty}\int_R (A_y w(Q))k_i(Q)dQ$$
$$=\int_R (A_y w(Q))\tilde{u}(Q,t)dQ=\int_R w(Q)(\tilde{A}_y\tilde{u}(Q,t))dQ$$

by (2.1) and by the definition (2.5) of $\tilde{A}$.

Lemma 4. We have, for $w(Q)\in L$ *and for* $1\leqq i\leqq q$,

(4.2) $$\int_R w(P)(A_y^{q-i}U_{2q}(P,Q))dP\in L.$$

Proof. By (3.16), we see that the integral vanishes outside a compact coordinate neighbourhood of P_0. Moreover, by (3.4), (3.15), (3.16) and (3.17), we see that the integral is twice continuously differentiable in Q (*Q.E.D.*).

We have, by (3.18),

$$C(P)k_i(P)=\int_R (A_y^{q-1}U_{2q}(P,Q))(A_y k_i(Q))dQ$$

in a certain vicinity of P_0. Let $w(Q)\in L$ vanish outside this vicinity. Letting $i\to\infty$ in

$$\int_R w(P)C(P)k_i(P)dP=\int_R w(P)dP\left\{\int_R (A_y^{q-1}U_{2q}(P,Q))(A_y k_i(Q))dQ\right\},$$

we obtain, by the Lemma 3 and Lemma 4,

(4.3) $$\tilde{u}(P,t)=C(P)^{-1}\int_R (A_y^{q-1}U_{2q}(P,Q))(\tilde{A}_y\tilde{u}(Q,t))dQ$$ almost everywhere in P in a certain vicinity of P_0.

The function $\tilde{u}(Q,t)$ belongs to $D(\tilde{A}_y^p)$ for every $p>0$. Thus we see, by the Lemma 3, that there exists a sequence of functions $\{k_i(Q)\}\subseteqq L$ such that

(4.4) $$\text{strong}\lim_{i\to\infty} k_i(Q)=\tilde{A}_y\tilde{u}(Q,t),$$

$$\lim_{i\to\infty}\int_R w(Q)(A_y k_i(Q))dQ=\int_R w(Q)(\tilde{A}_y^2\tilde{u}(Q,t))dQ \text{ for every } w(Q)\in L.$$

Hence we have

(4.5) $\int_R (A_y^{q-1}U_{2q}(P,Q))(\tilde{A}_y\tilde{u}(Q,t))dQ = \lim_{i\to\infty}\int_R (A_y^{q-1}U_{2q}(P,Q))k_i(Q)dQ$

almost everywhere in P. Also, by Green's integral theorem,

$$\begin{aligned}
&\int_R (A_y^{q-1}U_{2q}(P,Q))k_i(Q)dQ \\
&= \lim_{\kappa\to 0}\int_{R-\{Q;\Gamma(P,Q)\leqq\kappa\}} (A_y^{q-1}U_{2q}(P,Q))k_i(Q)dQ \\
&= \lim_{\kappa\to 0}\int_{R-\{Q;\Gamma(P,Q)\leqq\kappa\}} (A_y^{q-2}U_{2q}(P,Q))(A_yk_i(Q))dQ \\
&- \lim_{\kappa\to 0}\int_{\Gamma(P,Q)=\kappa}\left\{\frac{\partial A_y^{q-2}U_{2q}(P,Q)}{\partial\nu}k_i(Q) - (A_y^{q-2}U_{2q}(P,Q))\frac{\partial k_i(Q)}{\partial\nu}\right\}dS \\
&= \int_R (A_y^{q-2}U_{2q}(P,Q))(A_yk_i(Q))dQ.
\end{aligned}$$

The last equality may be obtained, as in the proof of (3.18), from the order relation (3.17):

$$A_y^{q-2}U_{2q}(P,Q) = 0(\Gamma(P,Q)^{(4-m)/2}).$$

Hence, for any $w(P)\in L$, we have

$$\begin{aligned}
&\int_R w(P)dP\left\{\int_R (A_y^{q-1}U_{2q}(P,Q))k_i(Q)dQ\right\} \\
&\qquad = \int_R w(P)dP\left\{\int_R (A_y^{q-2}U_{2q}(P,Q))(A_yk_i(Q))dQ\right\}.
\end{aligned}$$

Thus, by letting $i\to\infty$, we obtain, from (4.4), (4.5) and the Lemma 4,

$$\int_R (A_y^{q-1}U_{2q}(P,Q))(\tilde{A}_y\tilde{u}(Q,t))dQ = \int_R (A_y^{q-2}U_{2q}(P,Q))(\tilde{A}_y^2\tilde{u}(Q,t))dQ$$

almost everywhere in P. Repeating the process, we obtain, from (4.3),

Theorem 2. Let the dimension m be odd and $\geqq 2$, and let $2n\geqq m$ in the definition of $U_{2q}(P,Q)$. Then, for the initial data f and h in D, we have

(4.6) $\tilde{u}(P,t) = C(P)^{-1}\int_R U_{2q}(P,Q)(\tilde{A}_y^q\tilde{u}(Q,t))dQ$ *almost everywhere in P in a certain vicinity of P_0.*

Corollary. $\tilde{u}(Q,t)$ is, for fixed t, equal almost everywhere to a function $u(P,t)$ which is infinitely differentiable in P in a certain vicinity of P_0 such that

(4.6)′ $u(P,t) = C(P)^{-1}\int_R U_{2q}(P,Q)(\tilde{A}_y^q\tilde{u}(Q,t))dQ.$

Proof. We see that, if $q\geqq m$,

$$u(P,t) = C(P)^{-1}\int_R U_{2q}(P,Q)(\tilde{A}_y^q\tilde{u}(Q,t))dQ$$

is, by (3.17), q times continuously differentiable in P. As q may be taken arbitrarily large, the Corollary is proved.

In the above, we have assumed that the dimention m be odd and $\geqq 2$. Let us consider the case in which m does not satisfy this condition. In such a case, let $m'>m$ be odd and $\geqq 2$. We consider the function

$$\hat{u}(\hat{Q}, t)=u(y^1, \dots, y^m, t) \exp(-(y^{m+1})^2-\dots-(y^{m'})^2)$$

of m' independent variables $y^1,\dots, y^m, y^{m+1},\dots, y^{m'}$. By introducing the operator

$$(4.7) \qquad A^{(1)}=A+\frac{\partial^2}{\partial(y^{m+1})^2}+\dots+\frac{\partial^2}{\partial(y^{m'})^2}$$

in place of the operator $A=A_y$, we see, as above, that (4.6)$'$ holds good for $u(\hat{Q},t)$ in this case also. *Proof.* $\tilde{A}^{(1)q}\hat{u}(\hat{Q},t)$ belongs, for fixed t, to the product Hilbert space

$$L^a \times L_2(-\infty<y^{m+1}<\infty,\dots,-\infty<y^{m'}<\infty)$$

and hence we may apply the proof of the Theorem 2 above[4].

Next since $u(Q,t)$ belong to $D(\tilde{A}_y^p)$ for every $p>0$, it is easy to see, by (2.10), that

$$(4.8) \qquad (\partial_t\partial_t)^r\tilde{A}_y^q u(Q,t)=\tilde{A}_y^{q+r}u(Q,t) \text{ for every } r\geqq 0.$$

Thus we see, by (4.6)$'$, that $u(P,t)$ is, for fixed P, infinitely differentiable in t.

Moreover, since $u(Q,t)$ is infinitely differentiable in Q, we have

$$(4.9) \qquad \tilde{A}_y^{q+r}u(Q,t)=A_y^{q+r}u(Q,t) \text{ almost everywhere in } Q.$$

For, we have, by the definition (2.5) of $\tilde{A}$,

$$\int_R w(Q)(\tilde{A}_y^{q+r}u(Q,t))dQ=\int_R (A_y^{q+r}w(Q))u(Q,t)dQ=\int_R w(Q)(A_y^{q+r}u(Q,t))dQ,$$

when $w(Q)$ is infinitely differentiable and vanishes outside a compact set contained in the interior of R.

Therefore, in view of (2.11), we have proved finally the

Theorem 3. When f and h are in D, the function $\tilde{u}(x,t)$ given by (2.10) is (x,t)-almost everywhere equal to an infinitely differentiable function $u(x,t)$ satisfying (1.1)-(1.2).

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1) Cf. K. Yosida: On the integration of diffusion equations in Riemannian spaces, to appear in the Proc. Amer. Math. Soc.

2) See K. Friedrichs: Spektraltheorie halbbeschränkter Operatoren, Math. Ann. **109** (1934), 456-487. H. Fruedenthal: Über die Friedrichssche Fortsetzung halbbeschränkter Hermitescher Operatoren, Proc. Amsterdam Acad. **39** (1936), 832-833.

3) Suggested by M. Riesz: L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math. **81** (1948), 1-223. Cf. L. Schwartz: Théorie des distributions, I (1950), p. 47.

4) This argument may be called a method of descent. Cf. J. Hadamard: Le problème de Cauchy et les équations aux dérivees partielles linéaires hyperboliques, (1932), p. 287.