44. On The Interval Containing At Least One Prime Number.

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Bertrand-Tschebyschef's theorem (1852) is well-known for the interval *between* x *and* $2x$ *where* $x>1$, within which at least one prime number exists; this paper, however, enables us to reduce it up to *between* x *and* $6x/5$ *where* $x\geqq 25$. In conformity with *Ramanujan**, we establish the proof of our theorem upon the following fundamental formula: $T(x)=\sum_{m=1}^{\infty}\psi(x/m)=\log\Gamma([x]+1)$ where $\psi(x)=\sum_{m=1}^{\infty}\vartheta(\sqrt[m]{x})$ and $\vartheta(x)=\sum_{p\leqq x}\log p$.

Lemma 1. *When* $n>1$,

$$\frac{1}{n}T(x)-T\Big(\frac{x}{n}\Big)\geqq\frac{1}{n}\log\Gamma(x)-\log\Gamma\Big(\frac{x+n-1}{n}\Big)\qquad(x\geqq 1)$$

and

$$\frac{1}{n}T(x)-T\Big(\frac{x}{n}\Big)\leqq\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\Big(\frac{x+1}{n}\Big)\qquad(x\geqq n).$$

Proof. Since $\frac{\Gamma'}{\Gamma}(s)=\int_0^{\infty}\Big(\frac{e^{-t}}{t}-\frac{e^{-st}}{1-e^{-t}}\Big)dt$ when $s>0$,

$$\frac{\Gamma'}{\Gamma}(x)-\frac{\Gamma'}{\Gamma}\Big(\frac{x+n-1}{n}\Big)=\int_0^{\infty}\frac{1}{1-e^{-t}}\Big(e^{-\frac{x+n-1}{n}t}-e^{-xt}\Big)dt>0\qquad(x>1)$$

and

$$\frac{\Gamma'}{\Gamma}(x+1)-\frac{\Gamma'}{\Gamma}\Big(\frac{x+1}{n}\Big)=\int_0^{\infty}\frac{1}{1-e^{-t}}\Big(e^{-\frac{x+1}{n}t}-e^{-(x+1)t}\Big)dt>0\qquad(x>0),$$

that is to say, $\frac{1}{n}\log\Gamma(x)-\log\Gamma\Big(\frac{x+n-1}{n}\Big)$ and $\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\Big(\frac{x+1}{n}\Big)$ are increasing functions when $x\geqq 1$ and $x>0$ resp.

Hence we have

$$\begin{aligned}&\frac{1}{n}\log\Gamma(x)-\log\Gamma\Big(\frac{x+n-1}{n}\Big)\\&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\Big(\frac{[x]+n}{n}\Big)&&(x\geqq 1),\\&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\Big(\Big[\frac{x}{n}\Big]+1\Big)=\frac{1}{n}T(x)-T\Big(\frac{x}{n}\Big)\\&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\Big(\frac{[x]+1}{n}\Big)&&([x]\geqq n-1),\\&\leqq\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\Big(\frac{x+1}{n}\Big)&&(x>0);\end{aligned}$$

* S. Ramanujan: A Proof of Bertrand's postulate (Collected papers, 208-209).

then, removing the intermedia, we should obtain this lemma.

As the special case of Lemma 1, we have

$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{7}\right)-T\left(\frac{x}{43}\right)-T\left(\frac{x}{1806}\right)$$

$$=\frac{1}{2}T(x)-T\left(\frac{x}{2}\right)+\frac{1}{3}T(x)-T\left(\frac{x}{3}\right)+\frac{1}{7}T(x)-T\left(\frac{x}{7}\right)$$
$$+\frac{1}{43}T(x)-T\left(\frac{x}{43}\right)+\frac{1}{1806}T(x)-T\left(\frac{x}{1806}\right)$$
$$\leqq \log\Gamma(x+1)-\log\Gamma\left(\frac{x+1}{2}\right)-\log\Gamma\left(\frac{x+1}{3}\right)-\log\Gamma\left(\frac{x+1}{7}\right)$$
$$-\log\Gamma\left(\frac{x+1}{43}\right)-\log\Gamma\left(\frac{x+1}{1806}\right) \qquad (x\geqq 1806),$$

which is computed by *Stirling's formula*, $\log\Gamma(x)=\left(x-\frac{1}{2}\right)\log x - x+\log\sqrt{2\pi}+\frac{\theta}{12x}$ $(0<\theta<1)$, as follows:

$$<(x+1)\log(x+1)-\frac{x+1}{2}\log\frac{x+1}{2}-\frac{x+1}{3}\log\frac{x+1}{3}-\frac{x+1}{7}\log\frac{x+1}{7}$$
$$-\frac{x+1}{43}\log\frac{x+1}{43}-\frac{x+1}{1806}\log\frac{x+1}{1806}-\frac{1}{2}\left(\log(x+1)-\log\frac{x+1}{2}\right.$$
$$\left.-\log\frac{x+1}{3}-\log\frac{x+1}{7}-\log\frac{x+1}{43}-\log\frac{x+1}{1806}\right)$$
$$-(x+1)+\frac{x+1}{2}+\frac{x+1}{3}+\frac{x+1}{7}+\frac{x+1}{43}+\frac{x+1}{1806}-4\log\sqrt{2\pi}+\frac{1}{12(x+1)}$$
$$=(x+1)\left(\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{7}\log 7+\frac{1}{43}\log 43+\frac{1}{1806}\log 1806\right)$$
$$+2\log(x+1)-\log 1806-4\log\sqrt{2\pi}+\frac{1}{12(x+1)}$$
$$<1.0824x+2\log(x+1)-10+\frac{1}{12x}<1.0851x \qquad (x\geqq 2000). \quad (1)$$

Similarly we have also

$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{5}\right)+T\left(\frac{x}{30}\right)$$
$$\geqq \log\Gamma(x)-\log\Gamma\left(\frac{x+1}{2}\right)-\log\Gamma\left(\frac{x+2}{3}\right)-\log\Gamma\left(\frac{x+4}{5}\right)+\log\Gamma\left(\frac{x+1}{30}\right)$$
$$-\frac{1}{30}(\log\Gamma(x+1)-\log\Gamma(x)) \qquad (x\geqq 30),$$
$$>x\log x-\frac{x+1}{2}\log\frac{x+1}{2}-\frac{x+2}{3}\log\frac{x+2}{3}-\frac{x+4}{5}\log\frac{x+4}{5}$$
$$+\frac{x+1}{30}\log\frac{x+1}{30}-x+\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+4}{5}-\frac{x+1}{30}$$

$$-\frac{1}{2}\left(\log x-\log\frac{x+1}{2}-\log\frac{x+2}{3}-\log\frac{x+4}{5}+\log\frac{x+1}{30}\right)$$
$$-\log\sqrt{2\pi}-\frac{10}{12(x+1)}-\frac{1}{30}\log x$$
$$=\left\{-\frac{x}{2}\log\left(1+\frac{1}{x}\right)-\frac{x}{3}\log\left(1+\frac{2}{x}\right)-\frac{x}{5}\log\left(1+\frac{4}{x}\right)\right.$$
$$\left.+\frac{x}{30}\log\left(1+\frac{1}{x}\right)+\frac{1}{2}+\frac{2}{3}+\frac{4}{5}-\frac{1}{30}-\frac{5}{6(x+1)}\right\}$$
$$+(x-1)\left(\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5-\frac{1}{30}\log 30\right)$$
$$-\frac{8}{15}\log x-\frac{7}{15}\log(x+1)-\frac{1}{6}\log(x+2)-\frac{3}{10}\log(x+4)$$
$$+\frac{14}{15}\log 30-\log\sqrt{2\pi};$$

here, the sum of the terms inside the crooked brackets $>\frac{1}{2x}\left(\frac{1}{2}-\frac{1}{3x}\right)+\frac{4}{3x}\left(\frac{1}{2}-\frac{2}{3x}\right)+\frac{16}{5x}\left(\frac{1}{2}-\frac{4}{3x}\right)-\frac{1}{60x}-\frac{5}{6x}=\frac{1}{60x}\left(100-\frac{958}{3x}\right)>0$ when $x\geqq 4$, then

$$>(x-1)\left(\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5-\frac{1}{30}\log 30\right)-\log\left(x+\frac{1}{2}\right)$$
$$-\frac{7}{15}\log\left(x+\frac{8}{3}\right)+\frac{14}{15}\log 30-\log\sqrt{2\pi}$$
$$>0.9212x-\frac{22}{15}\log(x+2)+1.3>0.916x \qquad (x\geqq 2000). \tag{2}$$

Lemma 2. *Both the upper and lower bounds of $\psi(x)$ are given by the following:*

$$1.086x>\psi(x)>0.916x-2.318 \qquad (x>0).$$

Proof. We have

$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{6}\right)$$
$$=\psi(x)+\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{6m-1}\right)-2\psi\left(\frac{x}{6m}\right)+\psi\left(\frac{x}{6m+1}\right)\right)$$
$$\geqq\psi(x)-\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{6m}\right)-\psi\left(\frac{x}{6m+1}\right)\right),$$

then
$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{7}\right)-T\left(\frac{x}{42}\right)$$
$$\geqq\psi(x)+\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{6m+1}\right)-\psi\left(\frac{x}{7m}\right)-\psi\left(\frac{x}{42m}\right)\right)$$

$$\geqq \psi(x)-\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{42m}\right)-\psi\left(\frac{x}{42m+1}\right)\right),$$

and once more

$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{7}\right)-T\left(\frac{x}{43}\right)-T\left(\frac{x}{1806}\right)$$

$$\geqq \psi(x)+\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{42m+1}\right)-\psi\left(\frac{x}{43m}\right)-\psi\left(\frac{x}{1806m}\right)\right)$$

$$\geqq \psi(x)-\sum_{m=1}^{\infty}\left(\psi\left(\frac{x}{1806m}\right)-\psi\left(\frac{x}{1806m+1}\right)\right)\geqq\psi(x)-\psi\left(\frac{x}{1806}\right).$$

Hence, according to (1), we have $\psi(x)-\psi(x/1806)<1.0851x$ ($x \geqq 2000$); and since it is verifiable that this is true also when $0<x<2000$, let us write x, $x/1806$, $x/1806^2$, $x/1806^3$, … for x, and add them side by side, then we obtain

$$\psi(x)<1.0851\left(x+\frac{x}{1806}+\frac{x}{1806^2}+\frac{x}{1806^3}+\ \ldots\right)<1.086x\quad (x>0).$$

Next, we have

$$T(x)-T\left(\frac{x}{2}\right)-T\left(\frac{x}{3}\right)-T\left(\frac{x}{5}\right)+T\left(\frac{x}{30}\right)$$

$$=\psi(x)+\psi\left(\frac{x}{7}\right)+\psi\left(\frac{x}{11}\right)+\psi\left(\frac{x}{13}\right)+\psi\left(\frac{x}{17}\right)$$

$$+\psi\left(\frac{x}{19}\right)+\psi\left(\frac{x}{23}\right)+\psi\left(\frac{x}{29}\right)+\ \ldots\ *$$

$$-\psi\left(\frac{x}{6}\right)-\psi\left(\frac{x}{10}\right)-\psi\left(\frac{x}{12}\right)-\psi\left(\frac{x}{15}\right)-\psi\left(\frac{x}{18}\right)$$

$$-\psi\left(\frac{x}{20}\right)-\psi\left(\frac{x}{24}\right)-\psi\left(\frac{x}{30}\right)-\ \ldots\ *\quad \leqq\psi(x),$$

therefore, according to (2), we have $\psi(x)>0.916x$ ($x\geqq 2000$); and since it is also verifiable that $0.916x-2.318$, instead of $0.916x$, is less than $\psi(x)$ when $0<x<2000$, we obtain

$$\psi(x)>0.916x-2.318 \qquad (x>0).$$

Theorem. *There exists at least one prime number p such as:*

$$a_n\leqq x<p<\left(1+\frac{1}{n}\right)x \qquad \begin{pmatrix} n=1,\ 2,\ 3,\ \ 4,\ \ 5; \\ a_n=2,\ 8,\ 9,\ 24,\ 25,\ \textit{resp.}\end{pmatrix}$$

Proof. In order to prove $\vartheta\left(\frac{n+1}{n}x\right)-\vartheta(x)>0$ for the values of x as small as possible, let us use

* The denominator in every term after $\psi(x/29)$ or $-\psi(x/30)$ is congruent to some of preceding ones with respect to 30.

$$\psi(x)-\psi(\sqrt{x})-\psi(\sqrt[3]{x}) \geqq \vartheta(x) \geqq \psi(x)-\psi(\sqrt{x})-\psi(\sqrt[3]{x})-\psi(\sqrt[5]{x}),$$

and we have

$$\vartheta\Big(\frac{n+1}{n}x\Big)-\vartheta(x) \geqq \psi\Big(\frac{n+1}{n}x\Big)-\psi\Big(\sqrt{\frac{n+1}{n}x}\Big)-\psi\Big(\sqrt[3]{\frac{n+1}{n}x}\Big)-\psi\Big(\sqrt[5]{\frac{n+1}{n}x}\Big)$$
$$-\psi(x)+\psi(\sqrt{x})+\psi(\sqrt[3]{x}),$$

then, by Lemma 2,

$$>0.916\Big(\frac{n+1}{n}x+\sqrt{x}+\sqrt[3]{x}\Big)-6.954$$
$$-1.086\Big(x+\sqrt{\frac{n+1}{n}x}+\sqrt[3]{\frac{n+1}{n}x}+\sqrt[5]{\frac{n+1}{n}x}\Big)$$

which becomes positive when $n \leqq 5$ for sufficiently large values of x, that is to say

$$\vartheta(2x)-\vartheta(x)>0 \qquad (x \geqq 18),$$

$$\vartheta\Big(\frac{3}{2}x\Big)-\vartheta(x)>0 \qquad (x \geqq 48),$$

$$\vartheta\Big(\frac{4}{3}x\Big)-\vartheta(x)>0 \qquad (x \geqq 109),$$

$$\vartheta\Big(\frac{5}{4}x\Big)-\vartheta(x)>0 \qquad (x \geqq 293)$$

and

$$\vartheta\Big(\frac{6}{5}x\Big)-\vartheta(x)>0 \qquad (x \geqq 2103).$$

While there is surely at least one prime number between x and $(n+1)x/n$ when $2 \leqq x<18$, $8 \leqq x<48$, $9 \leqq x<109$, $24 \leqq x<293$ and $25 \leqq x<2103$ according as $n=1$, 2, 3, 4 and 5 resp.; our theorem is thus proved.