110. *Modulared Sequence Spaces.*

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A collection R of sequences of real numbers $(x_1, x_2, \dots)$ is called a *sequence space*, if R is a linear space, i.e. $R \ni (x_1, x_2, \dots)$, $(y_1, y_2, \dots)$ implies $R \ni (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots)$. For two sequence spaces R and S, if there is a sequence of positive numbers α_ν $(\nu = 1, 2, \dots)$ such that, putting $y_\nu = \alpha_\nu x_\nu$ $(\nu = 1, 2, \dots)$, we obtain a one-to-one corresponding between $(x_1, x_2, \dots) \in R$ and $(y_1, y_2, \dots) \in S$, then we shall say that R and S are *equivalent* to each other and write $R \cong S$.

For a sequence of positive numbers $p_\nu \geqq 1$ $(\nu = 1, 2, \dots)$, we see easily that the totality of sequences $(x_1, x_2, \dots)$ subject to the condition

$$\sum_{\nu=1}^{\infty} \frac{1}{p_\nu} | \alpha x_\nu |^{p_\nu} < +\infty \qquad \text{for some} \quad \alpha > 0,$$

constitutes a sequence space. This sequence space will be denoted by $l(p_1, p_2, \dots)$. Furthermore, putting

$$m(x) = \sum_{\nu=1}^{\infty} \frac{1}{p_\nu} | x_\nu |^{p_\nu} \quad \text{for} \quad x = (x_1, x_2, \dots),$$

we obtain a modular[1] m on $l(p_1, p_2, \dots)$, and putting

$$\| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}$$

we can introduce a norm[2] $\| x \|$ into $l(p_1, p_2, \dots)$. Then $l(p_1, p_2, \dots)$ is complete by this norm.[3] Therefore, if $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$, then we can find positive numbers α, β such that $\| x \| \leqq \alpha \| y \|$, $\| y \| \leqq \beta \| x \|$ for a just described one-to-one correspondence $l(p_1, p_2, \dots) \ni x \leftrightarrow y \in l(q_1, q_2, \dots)$.[4]

In this paper we shall prove the following theorems.

Theorem 1. *In order that* $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$, *it is necessary and sufficient that we have*

$$\sum_{\nu=1}^{\infty} \alpha^{\frac{p_\nu q_\nu}{|p_\nu - q_\nu|}} < +\infty \quad \textit{for some} \quad \alpha > 0.$$

Here we make use of the convention $\alpha^\infty = 0$.

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1) H. Nakano: Modulared semi-ordered linear spaces, Tokyo Math. Book Series, I (1950), § 35.

2) Ibid., Theorems 44.8 and 43.6.

3) Ibid., Theorems 40.6 and 40.9.

4) Ibid., Theorem 30.28.

Theorem 2. *If* $\lim_{\nu\to\infty} p_\nu = 1$, *then every weakly convergent series in* $l(p_1, p_2, \dots)$ *is strongly convergent.*

§ 1. Proof of Theorem 1.

Lemma 1. *For sequences of positive numbers* $\xi_\nu(\nu = 1, 2, \dots)$, *if* $\sum_{\nu=1}^{\infty} \xi_\nu^{p_\nu} < +\infty$ *implies* $\sum_{\nu=1}^{\infty} (\alpha\xi_\nu)^{q_\nu} < +\infty$ *for some* $\alpha > 0$, *and if* $\sum_{\nu=1}^{\infty} \xi_\nu^{q_\nu} < +\infty$ *implies* $\sum_{\nu=1}^{\infty} (\alpha\xi_\nu)^{p_\nu} < +\infty$ *for some* $\alpha > 0$, *then we have*

$$l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots).$$

Proof. According to the definition, we conclude easily from the assumption that $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$ by the correspondence

$$y_\nu = \frac{p_\nu^{\frac{1}{p_\nu}}}{q_\nu^{\frac{1}{q_\nu}}} x_\nu \quad (\nu = 1, 2, \dots),$$

$(x_1, x_2, \dots) \in l(p_1, p_2, \dots)$, $(y_1, y_2, \dots) \in l(q_1, q_2, \dots)$.

Lemma 2. *If* $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$ *and the sequence* $q_\nu\ (\nu = 1, 2, \dots)$ *is bounded, then* $\sum_{\nu=1}^{\infty} \xi_\nu^{p_\nu} < +\infty$ *implies* $\sum_{\nu=1}^{\infty} \xi_\nu^{q_\nu} < +\infty$.

Proof. There is by assumption a sequence of positive numbers $\alpha_\nu(\nu=1, 2, \dots)$ such that $\sum_{\nu=1}^{\infty} \frac{1}{p_\nu} |x_\nu|^{p_\nu} < +\infty$ implies $\sum_{\nu=1}^{\infty} \frac{1}{q_\nu} |\alpha_\nu x_\nu|^{q_\nu} < +\infty$, since $q_\nu\ (\nu = 1, 2, \dots)$ is bounded by assumption. Thus, if $\sum_{\nu=1}^{\infty} \xi_\nu^{p_\nu} < +\infty$ but $\sum_{\nu=1}^{\infty} \xi_\nu^{q_\nu} = +\infty$, then we have $\sum_{\nu=1}^{\infty} \frac{1}{q_\nu}(\alpha_\nu p_\nu^{\frac{1}{p_\nu}} \xi_\nu)^{q_\nu} < +\infty$, and hence, putting $q = \sup_{\nu=1,2,\dots} q_\nu$, we can sellect a partial sequence $\nu_\mu\ (\mu = 1, 2, \dots)$ such that $q\nu_\mu < p\nu_\mu$ and

$$\frac{1}{q_\nu}(\alpha_\nu p_\nu^{\frac{1}{p_\nu}})^{q_\nu} < \frac{1}{2^{2q\mu}} \qquad \text{for} \quad \nu = \nu_\mu .$$

Then, putting $x_\nu = p_\nu^{\frac{1}{p_\nu}} 2^\mu$ for $\nu = \nu_\mu$ and $x_\nu = 0$ for the other ν, we obtain

$$\sum_{\nu=1}^{\infty} \frac{1}{p_\nu}(\alpha x_\nu)^{p_\nu} = \sum_{\mu=1}^{\infty} (\alpha 2^\mu)^{p_{\nu_\mu}} = +\infty$$

for every positive number α, but

$$\sum_{\nu=1}^{\infty} \frac{1}{q_\nu}(\alpha_\nu x_\nu)^{q_\nu} < \sum_{\mu=1}^{\infty} \frac{1}{2^{2q\mu}} 2^{q\mu} = \sum_{\mu=1}^{\infty} \frac{1}{2^{q\mu}} < +\infty,$$

contradicting the assumption that $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$ by the correspondence $y_\nu = \alpha_\nu x_\nu\ (\nu = 1, 2, \dots)$.

Lemma 3. *If* $\sum_{\nu=1}^{\infty} \alpha^{\frac{p_\nu q_\nu}{|p_\nu - q_\nu|}} < +\infty$ *for some* $\alpha > 0$, *then we have* $l(p_1, p_2, \dots) \cong l(q_1, q_2, \dots)$.

Proof. We can assume that $\sum_{\nu=1}^{\infty}\alpha^{\frac{p_\nu q_\nu}{|p_\nu-q_\nu|}}<+\infty$ for a positive number $\alpha<1$. If $\sum_{\nu=1}^{\infty}\xi_\nu^{p_\nu}<+\infty$, then we have

$$\sum_{\nu=1}^{\infty}(\alpha\xi_\nu)^{q_\nu}=\sum_{\xi_\nu^{p_\nu-q_\nu}\geqq\alpha^{q_\nu}}(\alpha\xi_\nu)^{q_\nu}+\sum_{\xi_\nu^{p_\nu-q_\nu}<\alpha^{q_\nu}}(\alpha\xi_\nu)^{q_\nu}$$

$$\leqq\sum_{\xi_\nu^{p_\nu-q_\nu}\geqq\alpha^{q_\nu}}\xi_\nu^{p_\nu}+\sum_{\xi_\nu^{p_\nu-q_\nu}<\alpha^{q_\nu}}\alpha^{\frac{p_\nu q_\nu}{p_\nu-q_\nu}}<+\infty,$$

because we have $\xi_\nu<1$ except for a finite number of ν, and if $\xi_\nu<1$, $\xi_\nu^{p_\nu-q_\nu}<\alpha^{q_\nu}$, then we have $p_\nu>q_\nu$ and $(\alpha\xi_\nu)^{q_\nu}<\alpha^{\frac{p_\nu q_\nu}{p_\nu-q_\nu}}$. We also can prove likewise that $\sum_{\nu=1}^{\infty}\xi_\nu^{q_\nu}<+\infty$ implies $\sum_{\nu=1}^{\infty}(\alpha\xi_\nu)^{p_\nu}<+\infty$. Therefore we obtain $l(p_1, p_2, \ldots)\cong l(q_1, q_2, \ldots)$ by Lemma 1.

Lemma 4. *If $l(p_1, p_2, \ldots)\cong l(q_1, q_2, \ldots)$ and the sequence $q_\nu(\nu=1, 2, \ldots)$ is bounded, then we have*

$$\sum_{\nu=1}^{\infty}\alpha^{\frac{1}{|p_\nu-q_\nu|}}<+\infty \qquad \textit{for some}\quad \alpha>0.$$

Proof. Considering partial sequences, we recognize easily that we need only prove the case where $p_\nu>q_\nu\ (\nu=1, 2, \ldots)$. If $\sum_{\nu=1}^{\infty}\alpha^{\frac{1}{p_\nu-q_\nu}}=+\infty$ for every $\alpha>0$, then we can determine a partial sequence $\nu_\mu\ (\mu=1, 2, \ldots)$ such that

$$1\leqq\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1}\left(\frac{1}{2^\mu}\right)^{\frac{1}{p_\nu-q_\nu}}<2.$$

Then, putting $\xi_\nu=\left(\frac{1}{2^\mu}\right)^{\frac{1}{(p_\nu-q_\nu)q_\nu}}$ for $\nu_\mu\leqq\nu<\nu_{\mu+1}$, we have

$$\sum_{\nu=1}^{\infty}\xi_\nu^{q_\nu}=\sum_{\mu=1}^{\infty}\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1}\left(\frac{1}{2^\mu}\right)^{\frac{1}{p_\nu-q_\nu}}=+\infty,$$

$$\sum_{\nu=1}^{\infty}\xi_\nu^{p_\nu}=\sum_{\mu=1}^{\infty}\sum_{\mu=\nu_\mu}^{\nu_{\mu+1}-1}\left(\frac{1}{2^\mu}\right)^{\frac{1}{p_\nu-q_\nu}+\frac{1}{q_\nu}}$$

$$<\sum_{\mu=1}^{\infty}2\left(\frac{1}{2^\mu}\right)^{\frac{1}{q}}<+\infty$$

for $q=\sup_{\nu=1,2,\ldots}q_\nu$. Therefore we can not have $l(p_1, p_2, \ldots)\cong l(q_1, q_2, \ldots)$ by Lemma 2.

Lemma 5. *If $\frac{1}{p_\nu}+\frac{1}{p_\nu'}=1$, $\frac{1}{q_\nu}+\frac{1}{q_\nu'}=1$ $(\nu=1, 2, \ldots)$, then $l(p_1, p_2, \ldots.)\cong l(q_1, q_2, \ldots.)$ is equivalent to $l(p_1', p_2', \ldots.)\cong l(q_1', q_2', \ldots)$.*

Proof. $l(p'_1, p'_2, \ldots)$ is the conjugate space[5] of $l(p_1, p_2, \ldots)$, considering every $x' = (x'_1, x'_2, \ldots) \in l(p'_1, p'_2, \ldots)$ as a linear functional on $l(p_1, p_2, \ldots)$ by

$$x'(x) = \sum_{\nu=1}^{\infty} x'_\nu x_\nu \qquad \text{for} \quad x = (x_1, x_2, \ldots) \in l(p_1, p_2, \ldots).$$

Similarly $l(q'_1, q'_2, \ldots)$ is the conjugate space of $l(q_1, q_2, \ldots)$. Thus we obtain easily our assertion by the definition of the conjugate space.

Lemma 6. *If $l(p_1, p_2, \ldots) \cong l(q_1, q_2, \ldots)$, then we have*

$$\sum_{\nu=1}^{\infty} \alpha^{\frac{p_\nu q_\nu}{|p_\nu - q_\nu|}} < +\infty \quad \textit{for some} \quad \alpha > 0.$$

Proof. If one of sequences p_ν and q_ν $(\nu = 1, 2, \ldots)$ is bounded, then there is by Lemma 4 a positive number $\alpha < 1$ for which $\sum_{\nu=1}^{\infty} \alpha^{\frac{1}{|p_\nu - q_\nu|}} < +\infty$, and we have obviously $\alpha^{\frac{p_\nu q_\nu}{|p_\nu - q_\nu|}} \leqq \alpha^{\frac{1}{|p_\nu - q_\nu|}}$ $(\nu = 1, 2, \ldots)$. Thus, considering partial sequences, we recognize easily that we need only prove the case where $p_\nu \geqq q_\nu \geqq 2$ $(\nu = 1, 2, \ldots)$. In this case, putting $\frac{1}{p_\nu} + \frac{1}{p'_\nu} = 1$, $\frac{1}{q_\nu} + \frac{1}{q'_\nu} = 1$, we have $p'_\nu \leqq q'_\nu \leqq 2$ $(\nu = 1, 2, \ldots)$ and $l(p'_1, p'_2, \ldots) \cong l(q'_1, q'_2, \ldots)$ by Lemma 5. Therefore there is by Lemma 4 a positive number $\alpha < 1$ for which $\sum_{\nu=1}^{\infty} \alpha^{\frac{1}{q'_\nu - p'_\nu}} < +\infty$. Since $\frac{1}{q'_\nu - p'_\nu} = \frac{(p_\nu - 1)(q_\nu - 1)}{p_\nu - q_\nu}$, we obtain then

$$\sum_{\nu=1}^{\infty} \alpha^{\frac{p_\nu q_\nu}{p_\nu - q_\nu}} \leqq \sum_{\nu=1}^{\infty} \alpha^{\frac{(p_\nu - 1)(q_\nu - 1)}{p_\nu - q_\nu}} < +\infty.$$

§ 2. Proof of Theorem 2.

We assume firstly that $p_\nu > 1$ $(\nu = 1, 2, \ldots)$ and $\lim_{\nu \to \infty} p_\nu = 1$. If a sequence of sequences

$$x_\mu = (x_{\mu,1}, x_{\mu,2}, \ldots) \in l(p_1, p_2, \ldots) \quad (\mu = 1, 2, \ldots)$$

is weakly convergent to 0, then we have obviously $\lim_{\mu \to \infty} x_{\mu,\nu} = 0$ for every $\nu = 1, 2, \ldots$, and $\sup_{\mu = 1, 2, \ldots} \| x_\mu \| < +\infty$.[6] Thus we can suppose further that $\| x_\mu \| \leqq 1$ $(\mu = 1, 2, \ldots)$ and hence $m(x_\mu) \leqq 1$.[7] If there is a positive number ε for which $m(x_\mu) > \varepsilon$ $(\mu = 1, 2, \ldots)$, tnen we can find a partial sequence ν_μ $(\mu = 1, 2, \ldots)$ such that

5) Ibid., Theorem 54.14.

6) Ibid., Theorem 32.6.

7) Ibid., Theorem 40.12.

$$\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{p_\nu} | x_{\mu,\nu} |^{p_\nu} > \varepsilon ,$$

$$p_\nu \leqq 1 + \frac{1}{2^\mu} \qquad \text{for} \quad \nu \geqq \nu_\mu .$$

For $\frac{1}{p_\nu} + \frac{1}{p'_\nu} = 1$ $(\nu = 1, 2, \ldots)$, putting $y_\nu = 0$ for $\nu < \nu_1$ and

$$y_\nu = | x_{\mu,\nu} |^{\frac{p_\nu}{p_{\nu'}}} \qquad \text{for} \quad \nu_\mu \leqq \nu < \nu_{\mu+1} ,$$

we have then $p'_\nu \geqq 2^\mu + 1$ for $\nu \geqq \nu_\mu$ and

$$\sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{p'_\nu} y_\nu^{p_{\nu'}} \leqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{2^\mu+1} | x_{\mu,\nu} |^{p_\nu} \leqq \frac{1}{2^\mu} \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{p_\nu} | x_{\mu,\nu} |^{p_\nu}$$

$$\leqq \frac{1}{2^\mu} m(x_\mu) \leqq \frac{1}{2^\mu} .$$

We obtain thus $\sum_{\nu=1}^{\infty} \frac{1}{p'_\nu} y_\nu^{p_\nu'} < +\infty$ and hence $(y_1, y_2, \ldots) \in l(p'_1, p'_2, \ldots)$. However we have for every $\mu = 1, 2, \ldots$

$$\sum_{\nu=1}^{\infty} | x_{\mu,\nu} | y_\nu \geqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} | x_{\mu,\nu} |^{1+\frac{p_\nu}{p_{\nu'}}} \geqq \sum_{\nu=\nu_\mu}^{\nu_{\mu+1}-1} \frac{1}{p_\nu} | x_{\mu,\nu} |^{p_\nu} > \varepsilon ,$$

and this relation is impossible, because $l(p'_1, p'_2, \ldots)$ is the conjugate space of $l(p_1, p_2, \ldots)$ and

$$| x_\mu | = (| x_{\mu,1} |, | x_{\mu,2} |, \ldots) \qquad (\mu = 1, 2, \ldots)$$

also is weakly convergent to 0.[8] Therefore, considering partial sequences, we conclude easily $\lim_{\mu\to\infty} m(x_\mu) = 0$, and hence $\lim_{\mu\to\infty} \| x_\mu \| = 0$.[9]

For $x = (x_1, x_2, \ldots) \in l(1, 1, \ldots)$ we have obviously

$$\| x \| = m(x) = \sum_{\nu=1}^{\infty} | x_\nu | .$$

Thus, if $x_\mu = (x_{\mu,1}, x_{\mu,2}, \ldots) \in l(1, 1, \ldots)$ is weakly convergent to then we have $\lim_{\mu\to\infty} \| x_\mu \| = 0$.[10] Therefore, considering partial sequences, we conclude Theorem 2.

Finally we remark that, putting

$$p_\nu = 1 + \frac{1}{\log(\log(\nu+4))} \qquad (\nu = 1, 2, \ldots)$$

we have $\lim_{\nu\to\infty} p_\nu = 1$, but not $l(p_1, p_2, \ldots) \cong l(1, 1, \ldots)$ by Theorem 1, since we have for every $\alpha > 0$

$$\sum_{\nu=1}^{\infty} \alpha^{\frac{p_\nu}{p_\nu - 1}} = \sum_{\nu=1}^{\infty} \alpha (\log(\nu+4))^{\log \alpha} = +\infty .$$

8) H. Nakano: Discrete semi-ordered linear spaces (in Japanese), Functional Analysis, I (1947-9) 204-207. I. Halperin and H. Nakano: Discrete semi-ordered linear spaces, Canadian Jour. of Math., III (1951) 293-298, Lemma 1.

9) C.f. 1), Theorem 40.5.

10) J. Schur: Ueber lineare Transformationen in der Theorie der unendlichen Reihen, Jour. für reine und angew. Math. 151 (1921) 79-111. C. f. 8).