24. On the Theory of Semi-Local Rings.

By Masayoshi NAGATA.

(Comm. by Z. SUETUNA, M.J.A., May 12, 1950.)

Introduction.

The concept of local ring was introduced by Krull [7][1]. That of semi-local ring, a generalization of local ring, was introduced by Chevalley [1]. It was defined namely as a Noetherian ring R possessing only a finite number of maximal ideals. If $\mathfrak{m}$ denotes the intersection of all maximal ideals in a semi-local ring R, then $\bigcap_{n=1}^{\infty}\mathfrak{m}^n=(0)$, and so, R becomes a topological ring with $\{\mathfrak{m}^n\}$ as a system of neighbourhoods of zero. Chevalley derived many properties by making use of the concept of ring of quotients introduced by Grell [5]. He also introduced, in [2], a generalization of ring of quotients, in order to generalize Proposition 8, § II, [1]. But this generalization was only with respect to a Noetherian ring and the complementary set of a prime ideal. A further, and very natural, generalization of the concept of ring of quotients was given by Uzkov [6]. But it seems to me that also this generalization is not convenient to be applied to a generalized theory of semi-local rings which I want to present in the following. So we first introduce, after a short discussion of Uzkov's ring of quotients, a notion of topological quotient ring, which constitutes Chapter I. In Chapter II, we introduce semi-local rings in our generalized sense. They enjoy, besides some other properties, most of the propositions in [1]; an exception is the assertion that R is a complete semi-local ring with the intersection $\mathfrak{m}$ of all maximal ideals and if R' is a ring such as (1) R' contains R as a subring and (2) $\bigcap_{n=1}^{\infty}\mathfrak{m}R'=(0)$, then there exists $m(n)$ for each n such as $\mathfrak{m}^{m(n)}R'\cap R\subseteq\mathfrak{m}^n$ (a part of Proposition 4, II, 1). Appendix gives some supplementary remarks concerning our generalized notions.

We list the correspondences between the assertions in the present paper and those in [1, § II] or [3, Part I]:

Throughout this paper, a ring means a commutative ring with the identity element. Under a subring we mean a subring having the same identity. We will say that α is integral over a ring R if α satisfies a suitable monic equation with coefficients in R. θ denotes the empty set.

---

1) The number in brackets refers to the bibliography at the end.

Table

The present paper	Chevalley [1, § II]	Cohen [3, Part I]
Proposition 2	Theorem 1	Theorems 1, 2
Proposition 3	Proposition 6	
Proposition 4	Lemma 3	
Proposition 5	Proposition 2	
Proposition 6	Proposition 8	
Corollary to Lemma 2	Lemmas 4, 5	
Proposition 9	Proposition 3	The last part of Theorem 7
Proposition 10	Proposition 4	Corollary to Theorem 8
Propositions 11, 12	Proposition 7	
Proposition 13	Propositions 1, 5	
Proposition 16b		Lemma 4

Chapter I. Rings of Quotients[2].

1. $R_{\mathfrak{a}_S}$

Definition 1. Let R be a ring and S a subset of R closed under multiplication and not containing zero. Let $\mathfrak{a}$ be an ideal such as $S+\mathfrak{a}/\mathfrak{a}$ has no zero divisor in $R/\mathfrak{a}$. Then we denote by $R_{\mathfrak{a}_S}$ the ring of quotients of $S+\mathfrak{a}/\mathfrak{a}$ with respect to $R/\mathfrak{a}$. (Throughout this paper we maintain the meanings of R and S).

Definition 2. Let I be an ideal in R and I_S an ideal in $R_{\mathfrak{a}_S}$. Then we denote by $IR_{\mathfrak{a}_S}$ the ideal $\varphi(I)R_{\mathfrak{a}_S}$ in $R_{\mathfrak{a}_S}$ and by $I_S \supset R$ the ideal $\varphi^{-1}(I_S \cap R/\mathfrak{a})$, where φ is the natural homomorphism of R into $R/\mathfrak{a}$.

We see readily :

(1) $(I_S \cap R)R_{\mathfrak{a}_S}=I_S$ for every ideal I_S in $R_{\mathfrak{a}_S}$.

(2) $(I_{S1} \cap I_{S2}) \cap R=(I_{S1} \cap R) \cap (I_{S2} \cap R)$ for any two ideals I_{S1} and I_{S2} in $R_{\mathfrak{a}_S}$.

(3) Let $\mathfrak{p}$ be a prime ideal in R and $\mathfrak{q}$ a primary ideal belonging to $\mathfrak{p}$. Then (a) if $\mathfrak{p} \cap S \neq \theta$ we have $\mathfrak{q} \cap S \neq \theta$ and $\mathfrak{p}R_{\mathfrak{a}_S}=\mathfrak{q}R_{\mathfrak{a}_S}=R_{\mathfrak{a}_S}$; (b) if $\mathfrak{p} \cap S=\theta$ and $\mathfrak{q} \supseteq \mathfrak{a}$, $\mathfrak{q}R_{\mathfrak{a}_S}$ is a primary ideal belonging to $\mathfrak{p}R_{\mathfrak{a}_S}$, furthermore, $\mathfrak{p}R_{\mathfrak{a}_S} \cap R=\mathfrak{p}$ and $\mathfrak{q}R_{\mathfrak{a}_S} \cap R=\mathfrak{q}$; $\mathfrak{q}$ is strongly primary if and only if $\mathfrak{q}R_{\mathfrak{a}_S}$ is so.

(4) If $I=\bigcap_{\lambda \in \Lambda} \mathfrak{q}_\lambda$ is an intersection of primary ideals $\mathfrak{q}_\lambda$ in R and if $I \supseteq \mathfrak{a}$, we have $IR_{\mathfrak{a}_S}=\bigcap_{\lambda \in \Lambda} \mathfrak{q}_\lambda R_{\mathfrak{a}_S}$.

(5) If $I=\bigcap_{i=1}^{m} \mathfrak{q}_i$ is an intersection of primary ideals $\mathfrak{q}_i$ in R and if $\mathfrak{q}_i \supseteq \mathfrak{a}$ or $\mathfrak{q}_i \cap S \neq \theta$ for each i, we have $IR_{\mathfrak{a}_S}=\bigcap_{i=1}^{m} \mathfrak{q}_i R_{\mathfrak{a}_S}$. If the intersection $\bigcap_{i=1}^{m} \mathfrak{q}_i$ is irredundant, it gives again an irredundant intersection when the components $\mathfrak{q}_i R_{\mathfrak{a}_S}=R_{\mathfrak{a}_S}$ are omitted.

---

2) Except in the definition of topological kernel of R (Definition 5), we need not assume the existence of the identity in R, throughout this Chapter.

2. Rings of quotients (cf. [6]).

Definition 3. Let $U=\{a \in R;\ as=0$ for some $s \in S\}$. Then we call R_{US} the ring of quotients of S with respect to R, and denote it by R_S.

Lemma 1. U is an ideal and $S+U/U$ has no zero divisor in R/U.

(Proof) If $a, b \in U$, $as_1=0$, $bs_2=0$ for some $s_1, s_2 \in S$. Hence $(a+b)s_1s_2=0$, $s_1s_2 \in S$. It follows that U is an ideal. If $sx \equiv 0$ (mod. U) ($s \in S$, $x \in R$), we have $s'sx=0$ for some $s' \in S$. Therefore $x \in U$. This proves that $S+U/U$ has no zero divisor in R/U.

Remark 1. If $\mathfrak{q}$ is a primary ideal in R such as $\mathfrak{q} \cap S=\theta$, then we have $\mathfrak{q} \supseteq U$.

Remark 2. Every $R_{\mathfrak{a}S}$, with allowable $\mathfrak{a}$, is a homomorphic image of R_S.

3. Topological quotient rings.

Lemma 2. Let I be an ideal which does not meet S. Then there exists an ideal $\mathfrak{p}$ such as $\mathfrak{p} \supseteq I$, $\mathfrak{p} \cap S=\theta$ and every ideal properly containing $\mathfrak{p}$ meets S. $\mathfrak{p}$ is necessarily a prime ideal.

(Proof) The existence of $\mathfrak{p}$ can be proved by Zorn's Lemma, and $\mathfrak{p}$ is prime because S is closed under multiplication.

Definition 4. The ideal $\mathfrak{p}$ in Lemma 2 is called a maximal ideal with respect to S.

Definition 5. Let $\{\mathfrak{p}_\lambda;\ \lambda \in \Lambda\}$ be the totality of maximal ideals in R with respect to S. We call the intersection D_S of all strongly primary ideals belonging to some $\mathfrak{p}_\lambda (\lambda \in \Lambda)$ the topological kernel of S with respect to R. When $S=\{1\}$, we call D_S the topological kernel of R.

Lemma 4. Let D be an intersection of some primary ideals which do not meet S. Then $S+D/D$ has no zero divisor in R/D.

(Proof) Trivial.

Definition 6. Let D_S be the topological kernel of S with respect to R. Then we call $R_{D_S S}$ the topological quotient ring of S with respect to R, and denote it by $R_{[S]}$.

Note: When S is the complementary set of a prime ideal $\mathfrak{p}$, we use "of $\mathfrak{p}$" in place of "of S" and we use the notations $R_{\mathfrak{p}}$ and $R_{[\mathfrak{p}]}$ in place of R_S and $R_{[S]}$ respectively.

Chapter II. Semi-Local Rings.

1. Generalized semi-local rings.

Definition 1. A generalized semi-local ring is a ring whose topological kernel is (0). In any generalized semi-local ring R a topology can be introduced by taking ideals $\mathfrak{m}^{(1)}, \mathfrak{m}^{(2)}, \ldots$ to be neighbourhoods of zero, where $\mathfrak{m}^{(n)}$ is the intersection of all n-th power of maximal ideals. This is the natural topology of generalized semi-local ring.

Definition 2. A semi-local ring is a generalized semi-local ring which has only a finite number of maximal ideals.

Local rings, which were already defined in [8], may be defined as follows;

Definition 3. A local ring is a semi-local ring which has only one maximal ideal.

Proposition 1. A generalized semi-local ring R is a subring of the direct sum of $R_{[\mathfrak{p}_\lambda]}$ $(\lambda \in \Lambda)$ where $\{\mathfrak{p}_\lambda ; \lambda \in \Lambda\}$ is the totality of maximal ideals in R. If we introduce in the direct sum the strong topology of product space, then R becomes its subspace.

(Proof) Trivial.

Proposition 2. A generalized semi-local ring has a completion $\overline{R}$. $\overline{R}$ is again a generalized semi-local ring. If $\bar{\mathfrak{p}}_1$ and $\bar{\mathfrak{p}}_2$ are two distinct maximal ideals in $\overline{R}$, $\bar{\mathfrak{p}}_1 \cap R$ and $\bar{\mathfrak{p}}_2 \cap R$ are distinct maximal ideals in R. There exists an inclusion preserving one-to-one correspondence between all of closed ideals in R and some of closed ideals in $\overline{R}$; if $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ correspond to each other, $\bar{\mathfrak{a}} \cap R = \mathfrak{a}$ and the closure of $\mathfrak{a}\overline{R}$ in $\overline{R}$ is $\bar{\mathfrak{a}}$.

(Proof) This follows from the general theory of completion of topological ring.

Remark. If R is a semi-local ring, $\overline{R}$ is also a semi-local ring. If R is a local ring, $\overline{R}$ is also a local ring.

Proposition 3. Let $\overline{R}$ be the completion of a generalized semi-local ring R. If an element u of R is not a zero divisor in R and if every $u\mathfrak{m}^{(n)}$ is closed in R, it is not in $\overline{R}$ either.

(Proof) Let $uv=0$ $(v \in \overline{R})$. We take a sequence (v_n) such that $v - v_n \in \mathfrak{m}^{(n)}$. $uv_n \in u\mathfrak{m}^{(n)}$, and we have $v_n \in \mathfrak{m}^{(n)}$ because u is not a zero divisor in R. Hence $v=0$.

2. Semi-local rings.

Let, throughout this section, R be a semi-local ring and $\mathfrak{m}$ be the intersection of all maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ in R.

Proposition 4. Let $a_1, \cdots, a_h$ be h elements in R. Then the system $x \equiv a_i$ (mod. $\mathfrak{p}_i^n$) $(i=1, 2, \cdots, h)$ is solvable, and all the solutions are congruent modulo $\mathfrak{m}^n$.

(Proof) Let $\mathfrak{a}_i = \bigcap_{j \neq i} \mathfrak{p}_j$. Then $\mathfrak{a}_i^n + \mathfrak{p}_i^n = R$. Let $e_{i,n}$ be an element of $\mathfrak{a}_i^n$ such as $e_{i,n} \equiv 1$ (mod. $\mathfrak{p}_i^n$). With such $e_{i,n}$ $(i=1, 2, \cdots, h)$ we have that $x = \sum_{i=1}^{h} e_{i,n} a_i$ is a solution. If x' is another solution, we have $(x'-x) \sum_{i=1}^{h} e_{i,n} \equiv 0$ (mod. $\mathfrak{m}^n$). $\sum_{i=1}^{h} e_{i,n}$ is a unit, because $\sum_{i=1}^{h} e_{i,n} \equiv 1$ (mod. $\mathfrak{p}_j$) for every j $(j=1, 2, \cdots, h)$. Therefore $x'-x \equiv 0$ (mod. $\mathfrak{m}^n$).

Proposition 5. If R is complete, there exists a system of idempotent elements $\{\varepsilon_i ; i=1, 2, \cdots, h\}$ such as $\varepsilon_i \notin \mathfrak{p}_i$, $\varepsilon_i \in \mathfrak{p}_j$ if $i \neq j$,

$\sum_{i=1}^{h}\varepsilon_i=1$, $\varepsilon_i\varepsilon_j=0$ if $j\neq i$ and R_{ε_i} is isomorphic with $R_{[\mathfrak{p}_i]}=R_{\mathfrak{p}_i}$.

(Proof) Take $e_{i,n}$ in the proof of Proposition 3. The h sequences $(e_{i,n})$ $(i=1, 2, \dots, h)$ are convergent. Their limits ε_i fulfills our requirement.

Remark. This proposition shows that $R=R_{\varepsilon_1}+\cdots+R_{\varepsilon_h}$ (direct sum), R_{ε_i} being local ring with ε_i as identity, and R is also the product space of $R\varepsilon_i$.

Proposition 6. Let $\bar{R}$ be the completion of R. Then $\bar{R}_{\varepsilon_i}$ explained in Proposition 5 is isomorphic with the completion of $R_{[\mathfrak{p}_i]}$ where $\mathfrak{p}_i$ is the intersection of R and the maximal ideal which corresponds to ε_i.

(Proof) If we observe the fact that the kernel of natural homomorphism of R into $\bar{R}_{\varepsilon_i}$ is $\bigcap_{n=1}^{\infty}\mathfrak{p}_i^n$, Proposition 6 follows from Proposition 5.

Proposition 7[3]. A semi-local ring R is Noetherian if and only if (1) every ideal is closed and (2) every maximal ideal has a finite basis.

(Proof) If R is Noetherian and if $\mathfrak{a}$ is an ideal in R, $R/\mathfrak{a}$ is clearly semi-local. Therefore $\mathfrak{a}$ is closed. Converse follows from Propositions 2 and 5 and the fact that a complete local ring whose maximal ideal has a finite basis is Noetherian: [8, Corollary to Proposition 2], [3, Theorem 3].

We mention by the way also.

Proposition 8. A local ring R whose maximal ideal is principal ideal (x) is a Noetherian local ring.

(Proof) Observe the fact that every ideal but (0) is an ideal generated by x^n for some n.

3. Some further properties.

Lemma 1[4]. An element a is integral over a ring R if and only if there exists a ring R' such as (1) R' contains R as a subring, (2) R' is a finite R-module and (3) $R'\ni a$.

(Proof) If a is integral over R, $R'=R[a]$ satisfies three conditions above. Conversely, if R' is such a ring as above, we can set $R'=\sum_{i=1}^{h}Ry_i$ with $y_1=1$. Then we have $ay_i=\sum_{j=1}^{h}a_{ij}y_j$ $(a_{ij}\in R,\ i=1, 2, \cdots, h)$. If we set $f(a)=|a\delta_{ij}-a_{ij}|$, $f(a)$ is a monic polynomial on a with coefficients in R. We have $f(a)y_i=0$ $(i=1, 2, \cdots, h)$. Therefore $f(a)=0$.

3) We can exclude neither of these 2 conditions: It is clear that we cannot exclude the condition (1); the example in Appendix (2) of [8] shows that we cannot exclude the condition (2).

4) I owe this proof to Prof. G. Azumaya.

This being said, we shall also make use of the following lemma due to Cohen and Seidenberg (cf. Theorem 2, § 1, [4][5]).

Lemma 2. Let R' be integral over a ring R. Then for every prime ideal $\mathfrak{p}$ in R there exists a prime ideal $\mathfrak{P}$ in R' such as $\mathfrak{P}\cap R=\mathfrak{p}$.

Corollary. Let R' be a ring containing R as a subring and which is a finite R-module. Let $\mathfrak{a}$ be an ideal in R. Then $\mathfrak{a}R'\neq R'$.

Proposition 9.* Let R be a semi-local ring. Let R' be a ring containing R as a subring and finite over R. Then R' is a semi-local ring and R is a subspace of R'. If R is complete, R' is also complete.

(Proof) Let $\mathfrak{P}$ be a maximal ideal in R', $\mathfrak{P}\cap R$ is a maximal ideal in R. If $\mathfrak{p}$ is a maximal ideal in R, then $R'/\mathfrak{p}R'$ is a finite module over the field $R/\mathfrak{p}$. This shows that there exists only a finite number of (maximal) ideals in R', say $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ and that $(\mathfrak{P}_1\cdots\mathfrak{P}_r)^k \subseteq \mathfrak{p}R'$ for some k. This proves the first part of our assertion. Now, let R be complete. Let (v_n) $(n=1, 2, \cdots)$ be a convergent sequence in R'. We set $R'=\sum_{i=1}^{m} Ry_i$. Then we write $v_n-v_{n-1}=\sum_j u_{n,j}y_j$ where $u_{n,j}$ are elements of the intersection of all $m(n)$-th powers of maximal ideals with $m(n)\uparrow\infty$ and $v_0=0$. Then $(u_{n,j})$ $(n=1, 2, \cdots)$ $(j=1, 2, \cdots, m)$ are m convergent sequences in R. Let a_j be their limits respectively. Then $\sum_j a_jy_j$ is the limit of the sequence (v_n). This proves the second part of our assertion.

Proposition 10. Let R be a complete semi-local ring (with maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$). If R' is a ring which contains R as a subring in which $\bigcap_{n=1}^{\infty}\mathfrak{m}^nR'=(0)$ (where $\mathfrak{m}=\bigcap_{i=1}^{h}\mathfrak{p}_i$), then $\mathfrak{m}R'\cap R=\mathfrak{m}$. Furthermore, if $R'/\mathfrak{m}R'$ is a finite $R/\mathfrak{m}$-module, R' is a finite R-module, whence R' is also a complete semi-local ring by Proposition 9.

(Proof) It is clear that $\mathfrak{m}R'\cap R\supseteq\mathfrak{m}$. If $\mathfrak{m}R'\cap R\neq\mathfrak{m}$, there exists at least one maximal ideal, say $\mathfrak{p}_1$, such as $\mathfrak{p}_1R'=R'$. Then we have $\mathfrak{m}^nR'=(\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_h)^nR'$, contrary to our assumption. So necessarily $mR'\cap R=\mathfrak{m}$. Now we assume that $R'/\mathfrak{m}R'$ is a finite $R/\mathfrak{m}$-module. We set $R'/\mathfrak{m}R'=\sum_{i=1}^{d}(R/\mathfrak{m})v_i^*$ and choose for each i an element v_i from v_i^*. Let x be any element of R'. We construct d sequences $(x_{i,n})$ $(i=1, 2, \cdots, d;\ n=0, 1, \cdots)$ such as $x\equiv\sum_{i=1}^{d}x_{i,n}v_i$ (mod. $\mathfrak{m}^nR'$). We set $x_{i,0}=0$ for each i. If $x_{i,n}$ $(i=1, \cdots, d)$ are already defined, we write $x-\sum_i x_{i,n}v_i=\sum_{k=1}^{N}y_k\xi_k$ with $y_k\in R'$, $\xi_k\in\mathfrak{m}^n$. Then

---

5) The proof can be simplified if we make use of the notion of the rings of quotients.

* See Correction at the end.

we can write $y_k \equiv \sum_i y_{k,i} v_i$ (mod. $\mathfrak{m}R'$) ($y_{k,i} \in R$). We set $x_{i,n+1} = x_{i,n} + \sum_{k=1}^{N} y_{k,i} \xi_k$ ($i=1, \cdots, d$). Then each $(x_{i,n})$ is convergent in R; let x_i be its limit ($i=1, \cdots, d$), and set $x' = \sum_i x_i v_i$. Then $x' - x \in \mathfrak{m}^n R'$ for every n, namely, $x' = x$. Therefore $R' = \sum_i R v_i$.

Proposition 11. Let R and R' be two semi-local rings such that R' contains R as a subring and a subspace and is a finite R-module. Let $\bar{R}$ and $\bar{R}'$ be the completions of R and R' respectively. Then, if $R' = \sum_{i=1}^{k} R y_i$, $\bar{R}' = \sum_{i=1}^{k} \bar{R} y_i$ (up to an isomorphism).

(Proof) Since R is a subspace of R', R is also a subsapace of $\bar{R}'$. So we can consider $\bar{R}$ as the closure of R in $\bar{R}'$. Then our assertion follows from the fact that $\sum_i \bar{R} y_i$ is a complete semi-local ring.

Proposition 12. If we assume, besides the assumption in Proposition 11, that R has no zero divisor in $\bar{R}'$, we have, (1) if elements $x_1, \cdots, x_m$ of R' are linearly independent over R, they are so over $\bar{R}$, (2) if an element u of $\bar{R}$ is a zero divisor in $\bar{R}'$, it is already so in $\bar{R}$.

(Proof) We can assume without loss of generality that $x_1, \cdots, x_m$ is a maximal system of linearly independent elements. Then we can find an element c of R such that $cR' \subseteq \sum_{i=1}^{m} R x_i$ ($c \neq 0$). If $\sum_{i=1}^{m} u_i x_i = 0$ ($u_i \in \bar{R}$) we choose m sequences $(u_{i,n})$ ($i=1, \cdots, m$) such as $\lim u_{i,n} = u_i$ and $\sum_i c u_{i,n} x_i \in \sum_i \mathfrak{m}^n x_i$, namely, $\sum_i c u_{i,n} x_i = \sum_i a_{i,n} x_i$, $a_{i,n} \in \mathfrak{m}^n$, where $\mathfrak{m}$ is the intersection of all maximal ideals in R. Since $x_1, \cdots, x_m$ are linearly independent, we have $c u_{i,n} = a_{i,n}$, namely $c u_{i,n} \in \mathfrak{m}^n$, whence $c u_i = 0$ (for every i). We have $u_i = 0$ for every i. Let next an element u of $\bar{R}$ be not a zero divisor in $\bar{R}$. Assume $uv = 0$ ($v \in \bar{R}'$). We can write $cv = \sum_i a_i x_i$ ($a_i \in \bar{R}$). Hence, $\sum_i u a_i x_i = 0$ and therfore $u a_i = 0$ ($1 \leq i \leq m$). Then we have $a_i = 0$ ($1 \leq i \leq m$). So, $cv = 0$ and $v = 0$.

Proposition 13. Let $\mathfrak{q}$ be an ideal in a semi-local ring R. Then $R/\mathfrak{q}$ is again a semi-local ring if and only if $\mathfrak{q}$ is closed in R. Let, when this is the case, $\bar{\mathfrak{q}}$ be the closure of $\mathfrak{q}$ in the completion $\bar{R}$ of R. Then $\bar{R}/\bar{\mathfrak{q}}$ is the completion of $R/\mathfrak{q}$.

(Proof) The first part is evident, while the second follows from Proposition 2.

Proposition 14. Let R be a semi-local ring with maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ ($h > 1$). Then there exists an element u such as $u \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $u \notin \mathfrak{p}_j$ for $j > r$, where $0 < r < h$.

(Proof) Trivial.

Proposition 15. Let R be a semi-local ring with maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$. If R is a subdirect sum of $R_{[\mathfrak{p}_i]}$, R is the direct sum of $R_{[\mathfrak{p}_i]}$.

(Proof) When $h=1$, our assertion is trivial. We will assume that $h>1$ and our assertion holds for semi-local rings with $h-1$ maximal ideals. We set $R_{[\mathfrak{p}_i]}=R_i$. Then $\mathfrak{a}=R\cap(R_2+\cdots+R_h)$ is an ideal in R. Further, $R/\mathfrak{a}=R_1$ by natural mapping. Let u_1 be an element of R such as $u\notin\mathfrak{p}_1$, and $u_1\in\mathfrak{p}_j$ for any $j>1$. The residue class of u_1 module $\mathfrak{a}$ is a unit in R_1. Therefore if we write $u_1=v_1+\cdots+v_h$ $(v_i\in R_i)$, we can assume that $v_1=\varepsilon_1$ where ε_1 is the image of 1 in R_1 and it is true that $v_j\in\mathfrak{p}_jR_j$ for any $j>1$. Then $v_j\equiv\varepsilon_j$ (mod. $\mathfrak{a}$), where ε_j is the image of 1 in R, because $1=\varepsilon_1+\cdots+\varepsilon_h$. $u_2=1-u_1=\sum_{j=2}^{h}(\varepsilon_j-v_j)\in\mathfrak{a}$. u_2 is a unit in $R_2+\cdots+R_h$. Let b_1 be the inverse element of u_2 in $R_2+\cdots+R_h$. Then there exists an element $b=c_1+b_1\in R$, $c_1\in R$ for $R/R\cap R_1$ is a semi-local ring with $h-1$ maximal ideals. Then $bu_2=\varepsilon_2+\cdots+\varepsilon_h$. Therefore $1-(\varepsilon_2+\cdots+\varepsilon_h)=\varepsilon_1\in R$. Therefore $R_1\subseteq R$; $R/R_1=R_2+\cdots+R_h$. This proves our assertion.

It seems to me very likely that if a complete semi-local ring R' contains a (semi-local) ring R as a subring and is a finite R-module, then R is complete. But I have been able to prove only some special case as follows :

Lemma 3. Let R be a Noetherian semi-local ring having no zero divisor. If there exists a complete semi-local ring R' which contains R as a subring and is a finite R-module, then R is complete.

(Proof) The completion $\overline{R}$ of R is then a finite R-module. Let u be an element of $\overline{R}$. Then 1, u are linearly dependent over R, by Proposition 12. Therefore $\alpha u=\beta$ $(\alpha\neq 0)$ for some α, $\beta\in R$. Since R is Noetherian, αR is closed. Therefore $\alpha R\ni\beta$. Since α is not a zero divisor in $\overline{R}$ (by Proposition 3), $u\in R$.

Proposition 16a. Let R and R' be two semi-local rings such as (1) R is a direct sum of a finite nubmer of Noetherian semi-local rings, each of which has no zero divisor, (2) R' contains R as a subring and (3) R' is a finite R-module. Then R is complete if (and only if) R' is.

(Proof) This follows immediately from Lemma 3.

Proposition 16b. Let R and R' be two semi-local rings such as (1) R' contains R as a subring and (2) R' has a linearly independent basis $\{y_1=1, y_2, \cdots, y_r\}$ over R. Then R is closed in R'. Therefore R is complete if any only R' is.

(Proof) This follows readily from the fact that R is a subspace of R'.

Remark. If a ring R is a subring of a semi-local ring R' which is integral over R (or, as a special case, finite over R), then R is a semi-local ring.

Appendix.

Proposition 17. If D is the topological kernel of R_S, then $R_S/D=R_{[S]}$.

(Proof) Trivial.

Therefore (1) $R_{[S]}$ is a generalized semi-local ring and (2) if R_S is a generalized semi-local ring, $R_S=R_{[S]}$.

Proposition 18. Let R be a Noetherian ring. If the family of maximal ideals with respect to S is finite, $R_S=R_{[S]}$.

(Proof) Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be all the maximal ideals with respect to S. Then R_S is a Noetherian ring having no maximal ideals other than $\mathfrak{p}_1R_S, \cdots, \mathfrak{p}_hR_S$. Therefore R_S is a Noetherian semi-local ring.

Proposition 19. A necessary and sufficient condition for a ring R to be a subring of a generalized semi-local ring is that zero ideal is an intersection of some strongly primary ideals.

(Proof) If (0) is the intersection of strongly primary ideals $\mathfrak{q}_\lambda(\lambda \in \Lambda)$ belonging to $\mathfrak{p}_\lambda$ respectively, then R is a subring of the direct sum of all $R_{[\mathfrak{p}_\lambda]}$. Conversely, if R is a subring of a generalized semi-local ring R', (0) in R is an intersection of strongly primary ideals because (0) in R' is so.

Bibliography.

[1] C. Chevalley: On the theory of local rings, Ann. of Math. Vol. 44 (1943) pp. 690-708.

[2] ———: On the notion of ring of quotients of a prime ideal, Bull. Amer. Math. Soc. Vol. 50 (1944).

[3] I. S. Cohen: On the structure and ideal theory of complete local rings, Trans. Amer. Math. Vol. 59 (1946) pp. 54-106.

[4] I. S. Cohen and A. Seidenberg: Prime ideals and integral dependence, Bull. Amer. Math. Soc. Vol. 52 (1946) pp. 252-261.

[5] H. Grell: Beziehungen zwischen den Idealen verschiedener Ringe, Math. Ann. Vol. 97 (1927) pp. 490-523.

[6] A. I. Uzkov: On the rings of quotients of commutative rings, Mat. Sbornik N. S. 22 (64) (1948).

[7] W. Krull: Dimensionstheorie in Stellenringen, J. Reine Angew. Math. Vol. 179 (1938) pp. 204-226.

[8] M. Nagata: On the structure of complete local rings, forthcoming in Nagoya Mathematical Journal.

Correction.

Read Proposition 9 as follows:

Proposition 9. Let R be a semi-local ring and let R' be a ring containing R as a subring and which is a finite R-module. Then (I) R' possesses only a finite number of maximal ideals. (II) If there exist elements $c \in R$, $x_1, \dots, x_n \in R'$ such that c is not a zero divisor in R' and $x_0=1, x_1, \dots, x_n$ are linearly independent over R and that $cR' \subseteq \sum_{i=0}^{n} Rx_i$, then R is a semi-local ring. (III) If R' possesses a linearly independent module basis over R, R is a closed subspace of R' (by virtue of (II), R' is a semi-local ring). (IV) If R' is semi-local and if R is complete, then R' is also complete.