2. *Positive Definite Functions on Homogeneous Spaces.*

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§ 0. *Introduction.* Recently I. Gelfand and D. Raikov [3][1] have established an elegant theory of unitary representations of locally compact groups, which may be considered to correspond to Peter-Weyl's theory on compact groups. On the other hand, Peter-Weyl's theory was extended to the theory of harmonics on compact homogeneous spaces by H. Weyl [1] and E. Cartan [2]. The purpose of the present paper is to give a similar extension to Gelfand-Raikov's theory[2].

Let Ω be a homogeneous space with a locally compact group G of homeomorphisms. We always assume the following condition:

(*) *If* p_0 *is any fixed point of* Ω, *then the subgroup* $H = \{\sigma\,;\ \sigma \in G,\ \sigma p_0 = p_0\}$ *of* G *is compact.*

In § 1 of the present paper, we introduce some preliminary notions. In § 2, we discuss the correspondence between positive definite functions on Ω^2 and cyclic unitary representations, and show that so-called extreme positive definite functions correspond to irreducible unitary representations. We establish in § 3, the theorem concerning the topologies in the set of positive definite functions on Ω^2, and in § 4, the theorems of approximation of so-called invariant continuous functions on Ω^2 by means of linear combinations of elementary positive definite functions and the existence of sufficiently many irreducible unitary representations.

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1) Number in Literature at the end of this paper.

2) It is impossible for the present author to read the paper [3], but the papers on the same subject by R. Godement [4] and by H. Yoshizawa [5] have become available to him.

reduced to the case of *groups*; the author's original proofs were more complicated.

§1. *Preliminaries.* Let Ω be a homogeneous space with a transitive locally compact group G of homeomorphisms. We shall denote the points of Ω by p, q, s, t, and the elements of G by ρ, σ, τ, especially the unit element by e. Fix a point $p_0 \in \Omega$, and assume that (*) *the subgroup* $H = \{\sigma ; \sigma p_0 = p_0\}$ *of* G *is compact.* We shall consider a fixed triple $\{\Omega, G, p_0\}$ and denote by H_p the set $\{\rho ; \rho p_0 = p\}$ and by ρ_p any element of H_p; we easily see that

(1,1) $$H_{\sigma p} = \sigma H_p \qquad \text{for any } \sigma \in G.$$

For any $K \subseteq G$ and $A \subseteq \Omega$, we shall denote by KA the set $\{\sigma p ; \sigma \in K, p \in A\}$ $(\subseteq \Omega)$.

Let $\{V_\alpha ; \alpha \in \Lambda\}$ be a comlete system of conditionally compact neighbourhoods of e. Then the system $\{V_\alpha p ; p \in \Omega, \alpha \in \Lambda\}$ gives a uniform stsucture (see [7]) in Ω. We can consider that $\Omega = G/H$ and they are locally compact uniform spaces; then we can define a left-invariant Haar measure on G, and that on H such as the total measure of H is equal to one, and also a G-invariant measure on Ω (see [8] p. 10 and pp. 42–45); and we consider the product measur on $\Omega^2 = \Omega \times \Omega$. We shall use the notations $L^p(G)$, $L^p(\Omega)$ and $L^p(\Omega^2)$ $(1 \leqq p \leqq \infty)$ as usual.

Definttion 1. A triple $\{\mathfrak{H}, U(\sigma), \zeta\}$ of a Hilbert space $\mathfrak{H}$, a group $\{U(\sigma)\}$ of unitary operators on $\mathfrak{H}$ and a point $\zeta \in \mathfrak{H}$, is called *a unitary representation* (abbreviated to *u-representation*) of $\{\Omega, G, p_0\}$' if there exist

i) a strongly continuous mapping $(p \to \zeta_p)$ from the space Ω *into* the sphere $\mathfrak{S} = \{\xi ; \|\xi\| = \kappa\}$[3] in $\mathfrak{H}$ (κ: positive constant) such that $p_0 \to \zeta$; and

ii) a strongly continuous homomorphic mapping $(\sigma \to U(\sigma))$ from the group G *onto* the group $\{U(\sigma)\}$ such that $U(\sigma)\zeta_p = \zeta_{\sigma p}$ for any $\sigma \in G$, $p \in \Omega$.

A representation $\{\mathfrak{H}, U(\sigma), \zeta\}$ is said to be *cyclic*[4] if $\{U(\rho_p)\zeta ; p \in \Omega\}$ spans $\mathfrak{H}$, and to be *irreducible* if there exist no $U(\sigma)$-invariant proper subspace in $\mathfrak{H}$.

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3) $\|\cdot\|$ and $(\cdot,\cdot)$ denote respectively the norm and the inner product in the Hilbert space $\mathfrak{H}$.

4) It is called *simple* in [4] and [5], we call it *cyclic* following after Gelfand and Raikov.

Definition 2. A complex valued measurable function $h(p, q)$ on Ω^2 is called to be *invariant*, if it satisfies the condition:

(1,2) $h(\sigma p, \sigma q) = h(p, q)$ for almost all $\langle p, q\rangle \in \Omega^2$ and any $\sigma \in G$.

We shall denote by J the totality of invariant functions.

Definition 3. A complex valued function $f(p, q)$ on Ω^2 is called *positive definite* (abbreviated to *p. d.*), if $f \in L^\infty(\Omega^2)$ and satisfies the conditions (1, 2) and

$$\iint f(p, q)x(p)\overline{x(q)}dpdq \geqq 0 \qquad \text{for any } x \in L^1(\Omega). \tag{1,3}$$

We denote by $\mathbf{P}$ the totality of p. d. functions on Ω^2.

Corollary. If $f(p, q) \in \mathbf{P}$ is continuous on Ω^2, the condition (1,3) *is equivalent with the following one:*

$$\sum_{i,j} \alpha_i \bar{\alpha}_j f(p_i, p_j) \geqq 0 \tag{1,3'}$$

for any complex numbers $\alpha_1, \cdots, \alpha_n$ and arbitrary $p_1, \cdots, p_n \in \Omega$ and we have

$$f(p, p) \geqq 0,\ f(p, q) = \overline{f(q, p)} \text{ and } |f(p, q)| \leqq f(p, p), \tag{1,4}$$

where $f(p, p)$ is independent of p (by (1,2)).

The equivalence of (1,3) and (1,3′) is obtained by the same way as in [8] pp. 55–57 and (1,4) is easily obtained from (1,3′).

The following Lemma 1 and Theorem 1, which may be proved easily, give important examples of p. d. functions:

Lemma 1. *For any $\xi(\sigma) \in L^2(G)$, the function*

$$f(p,q) = \int_G d\rho \int_H \xi(\sigma^{-1}\rho_p^{-1}\rho)\, d\sigma \int_H \overline{\xi(\tau^{-1}\rho_q^{-1}\rho)} d\tau \text{ }^{5)} \tag{1,5}$$

is a continuous p. d. function on Ω^2.

Theorem 1. *If $\{\mathfrak{H}, U(\sigma), \zeta\}$ is a u-representation of $\{\Omega, G, p_0\}$, then $f(p, q) = (U(\rho_p)\zeta,\ U(\rho_q)\zeta) \equiv (\zeta_p,\ \zeta_q)$ is a continuous p. d. function on Ω^2.*

If $\xi(\sigma) \in L^1(G)$, then $\int_H \xi(\rho_p\sigma)d\sigma$ depends only on p and is independent of special $\rho_p \in H_p$, and we have

$$\int_\Omega dp \int_H \xi(\rho_p\sigma)d\sigma = \int_G \xi(\tau)d\tau \tag{1,6}$$

(see [8], pp. 43–45). For any function $x(p)$ on Ω we can define a function $\xi_x(\sigma)$ on G by $\xi_x(\sigma) = x(\sigma p_0)$; then $\xi_x(\rho_p) = x(p)$ for any

5) The right side of (1,5) depends only on p and q, and is independent of special $\rho_p \in H_q$, and $\rho_q \in H_q$, by the left-invariance of the Haar measure on H.

$\rho_p \in H_p$. From (1,6) and the fact that the total measure of H equals one, it is easy to show that

Lemma 2. For every $\xi(\sigma) \in L^1(G)$, the function

$$x_\xi(p) = \int_H \xi(\rho_p \sigma) d\sigma$$

belongs to $L^1(\Omega)$; conversely, for every $x(p) \in L^1(\Omega)$, the function

$$\xi_x(\sigma) = x(\sigma p_0)$$

belongs to $L^1(G)$; and $x(p) = x_{\xi_x}(p)$.

Now we shall prove the following

Lemma 3. In order that an invariant function $f(p, q)$ be p. d., it is necessary and sufficient that the function $\varphi_f(\sigma) = f(\sigma p_0, p_0)$ is a p.d. function on G (see [5] § 3).

Proof. From (1,6) and Lemma 2, we have the following two relations, from which this lemma is deduced at once: for any $\xi(\sigma) \in L^1(G)$

$$(1,7) \qquad \iint_{G^2} \varphi_f(\tau^{-1}\sigma)\xi(\sigma)\overline{\xi(\tau)}d\sigma d\tau$$

$$= \int_G \int_G f(\sigma p_0, \tau p_0)\xi(\sigma)\overline{\xi(\tau)}d\sigma d\tau$$

$$= \int_\Omega \int_\Omega dp\,dq \int_H \int_H f(\rho_p \sigma p_0, \rho_q \tau p_0)\xi(\rho_p \sigma)\overline{\xi(\rho_q \tau)}d\sigma d\tau$$

$$= \int_\Omega \int_\Omega f(p, q)dp\,dq \int_H \xi(\rho_p \sigma)d\sigma \int_H \overline{\xi(\rho_q \tau)}d\tau$$

$$= \iint_{\Omega^2} f(p, q)x_\xi(p)\overline{x_\xi(q)}dp\,dq\,;$$

and conversely for any $x(p) \in L^1(\Omega)$

$$(1,8) \qquad \iint_{\Omega^2} f(p, q)x(p)\overline{x(q)}dp\,dq = \iint_{\Omega^2} f(p, q)x_{\xi_x}(p)\overline{x_{\xi_x}(q)}dp\,dq$$

$$= \iint_{G^2} \varphi_f(\tau^{-1}\sigma)\xi_x(\sigma)\overline{\xi_x(\tau)}d\sigma d\tau \qquad \text{(from (1,7).)}$$

§ 2. *Positive definite functions and cyclic unitary representations.* Theorem 1 (§ 1) and the following two theorems show the correspondence between p. d. functions on Ω^2 and cyclic u-representations of $\{\Omega, G, p_0\}$. It is easy to show

Theorem 2. If $\{\mathfrak{H}, U(\sigma), \zeta\}$ and $\{\mathfrak{H}', U'(\sigma), \zeta'\}$ are cyclic u-representations and

$$(U(\rho_p)\zeta, U(\rho_p)\zeta) = U'(\rho_p)\zeta', U'(\rho_q)\zeta') \text{ for all } \langle p, q\rangle \in \Omega^2,$$

then the above two representations are mutually unitary equivalent.

We shall prove the following

Theorem 3. For every p. d. function $f(p, q)$ on Ω^2, there exists a cyclic u-representation $\{\mathfrak{H}, U(\sigma), \zeta\}$ such that

$$f(p, q) = (U(\rho_p)\zeta,\ U(\rho_q)\zeta) \text{ for almost every } \langle p, q\rangle \in \Omega^2.$$

Proof. Since $\varphi_f(\sigma) = f(\sigma p_0, p_0)$ is a p.d. function on G (by Lemma 3), there exists a cyclic u-representation $\{\mathfrak{H}, U(\sigma), \zeta\}$ of the group G (see §§3 and 4 of [5]) such that

$$\varphi_f(\sigma) = (U(\sigma)\zeta, \zeta) \text{ for almost every } \sigma \in G.$$

Hence, by the relation between the measure on Ω and that on G (see §1—and [8] p. 45), we have

$$f(p, q) = \varphi_f(\rho_q^{-1}\rho_p) = (U(\rho_q^{-1}\rho_p)\zeta, \zeta)$$
$$= (U(\rho_p)\zeta,\ U(\rho_q)\zeta) \text{ for almost every } \langle p, q\rangle \in \Omega^2.$$

In order to show that $U(\rho_p)\zeta$ depends only on p and is independent of special $\rho_p \in H_p$, it is sufficient to prove that $\tau \in H$ implies $U(\tau)\zeta = \zeta$. Since $\tau \in H$ implies $\tau p_0 = p_0$, we have

$$(U(\sigma)\zeta,\ U(\tau)\zeta) = f(\sigma p_0, \tau p_0) = f(\sigma p_0, p_0) = (U(\sigma)\zeta, \zeta)$$

for any $\sigma \in G$, $\tau \in H$; then since $\{U(\sigma)\zeta;\ \sigma \in G\}$ spans $\mathfrak{H}$[6], we have $U(\tau)\zeta = \zeta$; and $\{U(\rho_p)\zeta;\ p \in \Omega\}$ spans $\mathfrak{H}$, as every $\sigma \in G$ belongs to a certain H_p.

Therefore we can put $U(\rho_p)\zeta = \zeta_p$, then

$$U(\sigma)\zeta_p = U(\sigma)U(\rho_p)\zeta = U(\rho_{\sigma p})\zeta = \zeta_{\sigma p} \qquad \text{(by (1,1))}.$$

We shall now show that the mapping $p \to \zeta_p$ is strongly continuous. For any $p \in \Omega$ and any $\varepsilon > 0$, there exists a neighbourhood V_α of e such that $\sigma \in V_\alpha$ implies $\|U(\sigma)\zeta_p - \zeta_p\| < \varepsilon$[6]; then for any $q \in V_\alpha p$, we can write $q = \sigma p$ $(\sigma \in V_\alpha)$ and consequently $\zeta_q = U(\rho_{\sigma p})\zeta = U(\sigma)\zeta_p$; hence $q \in V_\alpha p$ implies

$$\|\zeta_q - \zeta_p\| = \|U(\sigma)\zeta_p - \zeta_p\| < \varepsilon.$$

By the definition of the mapping $p \to \zeta_p$, it is clear that p_0 corresponds to ζ and that $\|\zeta_p\| = \|\zeta\| = \sqrt{\operatorname{ess.sup}_{\sigma \in G} |\varphi_f(\sigma)|}$ for any any $p \in \Omega$. Thus $\{\mathfrak{H}, U(\sigma), \zeta\}$ satisfies all conditions in Definition 1;—Theorem 3 has been proved.

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6) See Theorem 3 in [5].

Corollary. Every p. d. function on Ω^2 coincides with a continuous one almost everywhere in Ω^2.

Denote by E_0 the totality of functions $z(p, q)$ on Ω^2 of the form $z(p, q) = x(p)\overline{x(q)}$; $x \in L^1(\Omega)$, and E—the *real* closed linear envelope of E_0 with respect to the norm $\|\cdot\|_1$ in $L^1(\Omega^2)$. Then E is a *real* Banach space and, as will easily be proved, $\mathbf{P}$ is a weakly closed subset of the *real* conjugate space E^* of E. Then, by the above Corollary, we can assume that every $f \in \mathbf{P}$ is continuous and $|f(p, q)| \leq \|f\|_\infty = f(p_0, p_0)$ (by Corollary of Definition 3).

Now $\mathbf{P}_0 = \{f;\ f \in \mathbf{P},\ \|f\|_\infty \leq 1\}$ is a bounded, convex and weakly closed subset of E^*. Hence accordiug to the theorem by M. Krein and D. Milman (for example, see [4] § 13), every $f \in \mathbf{P}_0$ is weakly approximated by convex combinations of extreme ones, where *an extreme point* means such a point of $\mathbf{P}_0$ that is not an inner point of the segment combining any pair of two points of $\mathbf{P}_0$. It is easy to see that every extreme $f \in \mathbf{P}_0$ is of norm one, except the zero element.

We establish in Theorems 4 and 5 the correspondence between irreducible u-representations and extreme p. d. functions.

Theorem 4. *If $\{\mathfrak{H}, U(\sigma), \zeta\}$ is an irreducible u-representation and $\|\zeta\| = 1$, then $f(p, q) = (U(\rho_p)\zeta, U(\rho_q)\zeta)$ is an extreme p. d. function.*

Proof. Suppose that

$$f(p, q) = f_1(p, q) + f_2(p, q) \qquad f_1, f_2 \in \mathbf{P}_0;$$

then $\varphi_f(\sigma) = f(\sigma p_0, p_0) = f_1(\sigma p_0, p_0) + f_2(\sigma p_0, p_0) = \varphi_{f_1}(\sigma) + \varphi_{f_2}(\sigma)$.
Since $\{\mathfrak{H}, U(\sigma), \zeta\}$ is the u-representation of G corresponding to φ_f in the sence of § 4 of [5] (see the proof of Theorem 3 in the present paper) and is irreducible, we can write by Theorem 4 of [5] that

$$\varphi_{f_1}(\sigma) = \lambda\varphi_f(\sigma),\ \varphi_{f_2}(\sigma) = (1-\lambda)\varphi_f(\sigma); \qquad 0 < \lambda < 1$$

(see also Lemma 3). Hence

$$f_1(p, q) = \lambda f(p, q),\ f_2(p, q) = (1-\lambda) f(p, q);\ 0 < \lambda < 1, \text{ q. e. d.}$$

Theorem 5. *If $\{\mathfrak{H}, U(\sigma), \zeta\}$ ($\|\zeta\| = 1$) is a cyclic u-representation and $f(p, q) = (U(\rho_p)\zeta, U(\rho_q)\zeta)$ is an extreme p. d. function in $\mathbf{P}_0$, then $\{\mathfrak{H}, U(\sigma), \zeta\}$ is irreducible.*

Proof. If there exists a projection P in $\mathfrak{H}$ which commutes with every $U(\sigma)$, then

$$f(p,q)=(U(\rho_p)\zeta,\ U(\rho_q)\zeta)$$
$$=(U(\rho_p)P\zeta,\ U(\rho_q)P\zeta)+(U(\rho_p)(I-P)\zeta,\ U(\rho_q)(I-P)\zeta).$$

and $f_1(p,q)=(U(\rho_p)P\zeta,\ U(\rho_q)P\zeta)$ and $f_2(p,q)=U(\rho_p)(I-P)\zeta,\ U(\rho_q)(I-P)\zeta)$ are also p. d. functions. Since $f(p,q)$ is extreme, it follows that

$(PU(\rho_p)\zeta,\ U(\rho_q)\zeta)=(U\rho_p)P\zeta,\ U(\rho_q)P\zeta)=\lambda(U(\rho_p)\zeta,\ U(\rho_q)\zeta)$; and since $\{U(\rho_p)\zeta;\ p\in\Omega\}$ spans $\mathfrak{H}$, we have $P=\lambda I$; hence $\lambda=0$ or 1, as P is a projection, q. e. d.

Definition 4. A function $f\in P$ is called *elementary*, if the corresponding u-representation is irreducible and $\|f\|_\infty=1$.

Then the following theorem is evident by Theorem 5 and the theorem by M. Krein and D. Milman:

Theorem 6. *Every $f(p,q)\in \mathbf{P}_0$ is approximated weakly (in E^*) by convex combinations of elementary p.d. functions.*

§3. *The weak convergence and the uniform convergence of p. d. functions.* In this paragraph, we shall show the equivalence of the two convergence in the set $\mathbf{P}_1=\{f;\ f\in\mathbf{P},\ \|f\|_\infty\equiv f(p_0,p_0)=1\}$, i.e. the equivalence of weak convergence in E^* and the uniform convergence on any compact subset of Ω^2. Concerning the set Π_1 of p. d. functions $\varphi(\sigma)$ on G such that $\varphi(e)=1$, the equivalence of the weak convergence in $L^1(G)^*$ and the uniform convergence on any compact subset of G is already established (for example, see [6]), and $\varphi_f(\sigma)=f(\sigma p_0,p_0)$ belongs to Π_1 if and only if $f\in\mathbf{P}_1$. To the purpose of this paragraph, therefore, it is sufficient to show the following two lemmas.

Lemma 4. *It is necessary and sufficient for $f(p,q)\in\mathbf{P}$ to converge to $f_0(p,q)\in\mathbf{P}$ weakly in E^*, that $\varphi_f(\sigma)$ converges to $\varphi_{f_0}(\sigma)$ weakly in $L^1(G)^*$.*

Proof. We define $\xi\cdot\eta(\sigma)$, $\xi^*(\sigma)(\xi,\eta\in L^1(G))$ and the approximate identity $\{e_\alpha(\sigma)\}$ of $L^1(G)$ as in [5][7]. Then by the properties

$$\lim_\alpha\|e_\alpha{}^*\cdot\xi-\xi\|_1=0\quad(\|\cdot\|_1 \text{ denote the norm in } L^1(G)),$$

$$4\eta^*\cdot\xi=(\xi+\eta)^*\cdot(\xi+\eta)-(\xi-\eta)^*\cdot(\xi-\eta)$$
$$+i(\xi+i\eta)^*\cdot(\xi+i\eta)-i(\xi-i\eta)^*\cdot(\xi-i\eta)$$

7) $\xi\cdot\eta(\sigma)=\int_G\xi(\tau)\eta(\tau^{-1}\sigma)d\tau$; $\xi^*(\sigma)=\overline{\xi(\sigma^{-1})}\Delta(\sigma)$, where $\Delta(\sigma)$ is the density of right-invariant Haar measure with respect to left-invariant Haar measure μ; $e_\alpha(\sigma)=C_{V_\alpha}(\sigma)/\mu(V_\alpha)$, where $C_{V_\alpha}(\rho)$ denotes the characteristic function of V_α.

and

$$\int\varphi(\sigma)\xi^*\cdot\xi(\sigma)d\sigma = \iint\varphi(\tau^{-1}\sigma)\xi(\sigma)\overline{\xi(\tau)}d\sigma d\tau$$

for $\xi, \eta \in L^1(G)$, the condition that $\varphi(\sigma)$ converges to $\varphi_0(\sigma)$ weakly in $L^1(G)^*$, is equivalent with the following one: for any $\xi \in L^1(G)$, $\iint\varphi(\tau^{-1}\sigma)\xi(\sigma)\overline{\xi(\tau)}d\sigma d\tau$ converges to $\iint\varphi_0(\tau^{-1}\sigma)\xi(\sigma)\overline{\xi(\tau)}d\sigma d\tau$. Hence this lemma is clear from Lemmas 2 and 3 (see (1,7) and 1,8)).

Lemma 5. *It is necessary and sufficient for $h(p, q) \in J$ to converge to $h_0(p, q) \in J$ uniformly on any compact subset of Ω^2, that $\varphi_h(\sigma)$ converges to $\varphi_{h_0}(\sigma)$ uniformly on any compact subset of* G (For the later application we show this lemma for functions $\in J(\supseteq \mathrm{P})$ instead of P).

Proof. i) Suppose that $h \in J$ converge to $h_0 \in J$ uniformly on any compact subset of Ω^2. Then for any compact set $K \subseteq G$, the set $F = \{\sigma p_0;\ \sigma \in K\} \subseteq \Omega$——and consequently the set $\hat{F} = \{\langle p, p_0\rangle;\ p \in F\} \subseteq \Omega^2$ is compact; hence, if $|h(p, q) - h_0(p, q)| < \varepsilon$ on $\hat{F}$ for $\varepsilon > 0$, then

$$|\varphi_h(\sigma) - \varphi_{h_0}(\sigma)| = |h(\sigma p_0, p_0) - h_0(\sigma p_0, p_0)| < \varepsilon \text{ for any } \sigma \in K.$$

ii) Conversely, suppose that φ_h converge to φ_{h_0} $(h, h_0 \in J)$ uniformly on any compact subset of G. For any compact set $F \subseteq \Omega^2$, there exist compact sets $F_1, F_2 \subseteq \Omega$ such that $\hat{F} \subseteq F_1 \times F_2 (\subseteq \Omega^2)$; since H is compact, the sets

$$K_1 = \bigvee_{p \in F_1} H_p \text{ and } K_2 = \bigvee_{q\ F_2} H_q$$

are compact, and consequently $K_2^{-1}K_1$ also is compact. Then since $\langle p, q\rangle \in F$ implies $\rho_q^{-1}\rho_p \in K_2^{-1}K_1$, it follows that $|\varphi_h(\sigma) - \varphi_{h_0}(\sigma)| < \varepsilon$ on $K_2^{-1}K_1$ implies

$$|h(p, q) - h_0(p, q)| = \varphi_h(\rho_p^{-1}\rho_p) - \varphi_{h_0}(\rho_q^{-1}\rho_p| < \varepsilon \text{ for any } \langle p, q\rangle \in \hat{F},$$

q.e.d.

Thus we obtain the following

Theorem 7. *In order that $f \in \mathrm{P}_1$ converge to $f_0 \in \mathrm{P}_1$ uniformly on any compact subset of Ω^2, it is necessary and sufficient that f converges to f_0 weakly in E^*.*

§ 4. *Theorems of approximation.* By Theorems 6 and 7, it is immediately obtained that

Theorem 8. *Every p.d. function*[8] *on Ω^2 is approximated, uni-*

8) In this paragraph too, we consider only continuous p. d. functions.

formly on any compact subset of Ω^2, *by linear combinations with positive coefficients of elementary p. d. functions.*

We shall denote by Γ the totality of functions on G which are constant on every coset $H_p(\in G/H)$ (i. e. every $\varphi(\sigma)\in\Gamma$ is considered as a function on G/H). Then,

Lemma 6. *For any p. d. function*[9] $\varphi(\sigma)\in\Gamma$, *there exists a p. d. function* $f(p, q)$ *on* Ω^2 *such that* $\varphi(\sigma) = \varphi_f(\sigma) \equiv f(\sigma p_0\ p_0)$.

Proof. There exists a cyclic u-representation $\{\mathfrak{H}, U(\sigma), \zeta\}$ of the group G (see §§ 3 and 4 of [5]) such that $\varphi(\sigma) = (U(\sigma)\zeta, \zeta)$. Since $\varphi(\sigma)\in\Gamma$, $(U(\tau)\zeta, \zeta) = \varphi(\tau) = \varphi(e) = (\zeta, \zeta)$ for any $\tau\in H$; hence $(U(\tau)\zeta-\zeta,\ U(\tau)\zeta-\zeta) = 0$ (from $\| U(\tau)\zeta\| = \|\zeta\|$), i. e. $U(\tau)\zeta = \zeta$. Therefore we can show, as in the proof of Theorem 3, that $\{\mathfrak{H}, U(\sigma),\zeta\}$ is a u-representation of $\{\Omega, G, p_0\}$. Then

$$f(p, q) = (U(\rho_p)\zeta,\ U(\rho_q)\zeta) \equiv (\zeta_p, \zeta_q)$$

is a p. d. function on Ω^2 (Theorem 1), and

$$\varphi(\sigma) = (U(\sigma)\zeta, \zeta) = (\zeta_{\sigma r_0}, \zeta) = f(\sigma p_0, p_0), \text{ q.e.d.}$$

Theorem 9. *Every invariant continuous function on* Ω^2 *is approximated, uniformly on any compact subset of* Ω^2, *by linear combinations of elementary p. d. functions.*

Proof. For any invariant continuous function $h(p,q)$, the function $\varphi_h(\sigma) = h(\sigma p_0, p)$ is approximated by linear combinations of p. d. functions on G, uniformly on any compact subset of G (see [3] or [4]): for any compact set $K\subseteq G$ and any $\varepsilon>0$, there exists a linear combination $\psi(\sigma)$ of p.d. functions on G such that $|\varphi_h(\rho)-\psi(\rho)|<\varepsilon$ for $\rho\in HKH$ (HKH is compact as well as H and K). Then since $\varphi_h(\rho) = h(\rho p_0, p_0) = h(\sigma p_0, \tau p_0) = \varphi_h(\tau^{-1}\rho\sigma)$ for any $\sigma, \tau\in H$, we have

$$(4,1)\qquad |\varphi_h(\rho)-\iint_{H^2}\psi(\tau^{-1}\rho\sigma)d\sigma d\tau|$$
$$\leqq\iint_{H^2}|\varphi_h(\tau^{-1}\rho\sigma)-\psi(\tau^{-1}\rho\sigma)|\,d\sigma d\tau \quad \text{for any } \rho\in K.$$

On the other hand, for any p. d. function $\varphi(\rho)$ on G, $\varphi_1(\rho) = \iint_{H^2}\varphi(\tau^{-1}\rho\sigma)d\sigma d\tau$ is a p.d. function belonging to Γ; and by Lemma 6, $\varphi_1(\rho) = f(\rho p_0, p_0)$ for a certain $f\in \mathrm{P}$. Hence the function $\psi_1(\rho) = \iint_{H^2}\psi(\tau^{-1}\rho\sigma)d\sigma d\tau$ (in the left side of (4,1)) is expressed by

9) We can assume that every p. d. function on G is continuous; see [5] § 4.

$$(4,2) \qquad \psi_1(\rho) = \sum_{i=1}^{n} a_i \varphi_{f_i}(\rho) = \sum_{i=1}^{n} a_i f_i(\rho p_0, p_0)$$

(a_i: complex number, $f_i \in \mathrm{P}$). From (4,1), (4,2) and Lemma 5 (§ 3), we can see that $h(p, q)$ is approximated by linear combinations of p. d. functions on Ω^2 uniformly on any compact subset of Ω^2; and this theorem becomes clear by Theorem 8.

Theorem 10. i) *Let* $p \neq q$ *be two points of* Ω; *then there exists an irredusible u-representation* $\{\mathfrak{H}, U(\sigma), \zeta\}$ *such that* $\zeta_p \neq \zeta_q$.

ii) *If* τ *is an element of* G *different from* e, *then there exists an irreducible u-representation* $\{\mathfrak{H}, U(\sigma), \zeta\}$ *such that* $U(\tau) \neq I$.

Proof. i) Since H_p and H_q are compact as well as H and since $H_p \wedge H_q$ is empty (from $p \neq q$), there exists a neighbourhood V_α of e such that $H_p V_\alpha V_\alpha^{-1} H \wedge H_q$ is empty. Let $\xi(\rho)$ be the characteristic function of V_α, then

$$f(s, t) = \int_G d\rho \int_H \xi(\sigma^{-1}\rho_s^{-1}\rho) d\sigma \int_H \xi(\tau^{-1}\rho_t^{-1}\rho) d\tau$$

is a p.d. function on Ω^2 (Lemma 1), and $f(p, p) \neq 0 = f(q, p)$ ($f(p, p) \neq 0$ is clear; if $f(q, p) \neq 0$, then $\sigma^{-1}\rho_q^{-1}\rho \in V_\alpha$ and $\tau^{-1}\rho_q^{-1}\rho \in V_\alpha$ for some σ, $\tau \in H$ and $\rho \in G$, hence $\rho_q \in \rho V_\alpha^{-1}\tau \subseteq \rho_p \sigma V_\alpha V_\alpha^{-1}\tau \subseteq H_p V_\alpha V_\alpha^{-1} H$, consequently $H_p V_\alpha V_\alpha^{-1} H \wedge H_q$ is not empty — contradiction). Hence by Theorem 8, there exists an elementary p.d. function $f_0(s, t)$ such that $f_0(p, p) \neq f_0(q, p)$.

Suppose that $\zeta_p = \zeta_q$ for every irreducible u-representation, then for every elementary p.d. function $f_1(s, t)$, by the results of § 2, we have

$$f_1(p, p) = (\zeta_p, \zeta_p) = (\zeta_q, \zeta_p) = f_1(q, p),$$

which is a contradiction.

ii) From $\tau \neq e$, there exists a point $p \in \Omega$ such that $\tau p \neq p$, and by i), there exists an irreducible u-representation $\{\mathfrak{H}, U(\sigma), \zeta\}$ such that $\zeta_{\tau p} \neq \zeta_p$, hence

$$U(\tau)\zeta_p = \zeta_{\tau p} \neq \zeta_p,$$

namely $U(\tau) \neq I$, q.e.d.

Appendix.

If we assume, in the results of this paper, that G is — and consequently Ω also is — compact, we obtain the results of [1].

The method is as follows:

I) Let $\{\mathfrak{H}, U(\sigma), \zeta\}$ be an arbitrary u-representation of $\{\Omega, G, p_0\}$ and let the continuous function

$$\xi(p) = (\xi, U(\rho_p)\zeta)$$

on Ω correspond to the element $\xi \in \mathfrak{H}$. Then we can prove that $(U(\sigma)\xi)(p) = \xi(\sigma^{-1}p)$ and that, if $\{\mathfrak{H}, U(\sigma), \zeta\}$ is irreducible, there exists a $\lambda > 0$ such that

$$\lambda(\xi, \eta) = \int \xi(p)\overline{\eta(p)}dp \qquad \text{for any } \xi, \eta \in \mathfrak{H},$$

and $\mathfrak{H}$ is finite-dimensional, and let $\{\varphi_1, \cdots, \varphi_n\}$ be a complete orthonormal system, then $\xi = \sum a_i \varphi_i (\in \mathfrak{H})$ implies

$$\xi(p) = \sum a_i \varphi_i(p)^{10)}.$$

For any u-representation $\{\mathfrak{H}, U(\sigma), \zeta\}$, the corresponding p.d. function $f(p, q) = (U(\rho_p)\zeta, U(\rho_q)\zeta)$ is expressed by the seriese (with positive coefficients) of elementary p.d. functions (Cf. [4] § 24, Theorem 16 (4)), and $\{\mathfrak{H}, U(\sigma), \zeta\}$ is decomposable into the direct sum of countable number of irreducible u-representations. Hence every $\xi(p)$ $(\xi \in \mathfrak{H})$ is expressed by

$$\xi(p) = \sum_{\nu=1}^{\infty}\sum_{i=1}^{n_\nu} a_i^{(\nu)}\varphi_i^{(\nu)}(p), \tag{1}$$

where every $\{\varphi_1^{(\nu)}, \cdots, \varphi_{n_\nu}^{(\nu)}\}$ is a complete orthonormal system of $\mathfrak{H}^{(\nu)}$ in the irreducible u-representation $\{\mathfrak{H}^{(\nu)}, U^{(\nu)}(\sigma), \zeta^{(\nu)}\}$ and the series of the right side of (1) converges absolutely and uniformly on Ω.

II) Let R be the totality of linear combinations of the functions $\varphi_i^{(\nu)}(p)$ (defined above) and their uniform limit on Ω. We shall show that R is a ring; to this purpose, it is sufficient to prove that the product of two functions $\varphi_i^{(\mu)}(p)$ and $\varphi_j^{(\nu)}(p)$ also belongs to R, but it will easily be seen by Theorem 13 of [4], Lemma 3 in the present paper and the above equality (1).

It is evident that R contains the function $\varphi_0(p) \equiv 1$ and that $\psi(p) \in R$ implies $\overline{\psi(p)} \in R$. For any two points $p, q \in \Omega$, there exists a function $\psi(s) \in R$ such that $\psi(p) \neq \psi(q)$ — this fact is proved from the existence of sufficiently many irreducible u-representations (see Theorem 10 (i)).

10) The system $\{\varphi_1(p), \ldots, \varphi_n(p)\}$ spans a primitive harmonic set defined in [1].

III) Thus, by the well known theorem by I. Gelfand and G. Šilov, the ring R is the totality of all continuous functions on the compact space Ω; i.e. *an arbitrary continuous function on Ω is approximated uniformly by linear combinations of members of primitive harmonic sets on Ω.*

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