68. Trigonometrical Interpolation.

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1. Let $f(x)$ be a continuous function. It is well known that there is just one trigonometrical polynomial of order n which coincides with $f(x)$ in the points:

$$x_i = i\frac{2\pi}{2n+1}\ (i = 0, 1, 2 \dots, 2n).$$

If we write

$$\varphi_n(t) = x_i\ (i = 0, 1, 2, \dots, 2n)$$

for $x_i \leqq t < x_{i+1}$, the desired polynomial becomes

$$U_n(f, x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\,d\varphi_n(t)$$
$$= \frac{1}{2n+1}\sum_{i=0}^{2n} f(x_i)\frac{\sin(n+\frac{1}{2})(x-x_i)}{\sin\frac{1}{2}(x-x_i)}.$$

J. Marcinkiewicz [1] has given a continuous function $f(x)$ which does not satisfy

(1) $$\sum_{n=1}^{N} |U_n(f, x) - f(x)| = O(N)$$

everywhere. In §2, we prove that (1) holds almost everywhere for $f(x)$ satisfying a certain continuity condition. In §3, we prove inequality theorems concerning interpolation polynomials which are the analogon of the theorems due to Littlewood, Paley and Zygmund concerning Fourier series. The special case $p=2$ of our theorems are already proved by J. Marcinkiewicz [2, 3].

2. If we denote

$$\omega(h) = \operatorname*{Max}_{0\leqq x\leqq 2\pi} |f(x+h) - f(x)|$$

and

$$\theta_n(x) = \frac{1}{2}U_n(f, x) + \frac{1}{4}U_n\left(f, x+\frac{2\pi}{2n+1}\right) + \frac{1}{4}U_n\left(f, x-\frac{2\pi}{2n+1}\right),$$

then we get

Theorem 1. *If* $\sum_{n=1}^{\infty}\frac{1}{n}\left[\omega\left(\frac{1}{n}\right)\right]^p < \infty$, *where* $\infty > p \geqq 1$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n}|U_n(f, x) - \theta_n(x)|^p$$

converges almost everywhere. Accordingly under the same hypothesis

$$\sum_{n=1}^{N}|U_n(f, x) - f(x)|^p\,dx = o(N)$$

almost everywhere.

For the proof we begin with some lemmas.

Lemma 1. *If $f(x)$ is continuous, then*

$$\left(\int_0^{2\pi} | U_n(f,x)|^p\, dx\right)^{\frac{1}{p}} \leqq A_p \underset{0\leqq x\leqq 2\pi}{\mathrm{Max}} |f(x)|.$$

This is due to J. Marcinkiewicz [1].

Lemma 2. *If $f(x)$ is continuous, then $\theta_n(x)$ converges uniformly to $f(x)$.*

This is due to A. C. Offord [6].

We will now prove the theorem. By Lemma 1, we have

$$\int_0^{2\pi} | U_n(f,x)-\theta_n(x)|^p\, dx \leqq A_p \Big(\underset{0\leqq x\leqq 2\pi}{\mathrm{Max}} \Big|f(x)-\frac{1}{2}f\left(x+\frac{2\pi}{2\pi+1}\right)-\frac{1}{2} f\left(x-\frac{2\pi}{2n+1}\right)\Big|^p\Big) \leqq B_p \left[\omega\left(\frac{1}{n}\right)\right]^p,$$

and then

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left| U_n(f,x)-\theta_n(x)\right|^p dx \leqq B_p \sum_{n=1}^{\infty}\frac{1}{n}\left[\omega\left(\frac{1}{n}\right)\right]^p.$$

Since the right-hand side series converges, the integrand of the left-hand side is finite almost everywhere. By Kronecker's theorem

$$\sum_{n=1}^{N} | U_n(f,x)-\theta(x)|^p = o(N)$$

almost everywhere. In view of Lemma 2, we get

$$\sum_{n=1}^{N} | U_n(f,x)-f(x)|^p = o(N)$$

almost everywhere. Thus the theorem is proved.

Theorem 2. *If $\sum_{k=1}^{\infty}\left[\omega\left(\frac{1}{n_k}\right)\right]^p$ converges, $U_{n_k}(f,x)$ converges to $f(x)$ almost everywhere.*

Proof is immediate by the fact

$$\int_0^{2\pi} \sum_{k=1}^{\infty} | U_{n_k}(f,x)-\theta_{n_k}(x)|^p\, dx \leqq C_p \sum_{k=1}^{\infty}\left[\omega\left(\frac{1}{n_k}\right)\right]^p.$$

3. We denote by A^p the class of absolutely continuous function $f(x)$ with $f(0)=f(2\pi)=0$ and $f'(x) \varepsilon L^p$; by $U'_n(f,x)$ the derivative of $U_n(f,x)$ and lastly by $\|f\|_p$ the expression

$$\left(\int_0^{2\pi} |f|^p\, dx\right)^{\frac{1}{p}}.$$

After Marcinkiewicz-Zygmund [4] we denote the interpolation polynomial by $\sum^{n} c_\nu^{(n)} e^{i\nu x}$. Further we put

$$U_{n,i}(f,x) = \sum_{\nu=1}^{i} c_\nu^{(n)} e^{i\nu x},$$

and

$$\sigma'_n(f,x) = (\sum_{i=1}^{n} U'_{n,i}(f,x)) / n,$$

then we have

Theorem 3. *If* $f(x) \varepsilon A^p$ $(p>1)$,

$$\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|U'_n(f,x)-\sigma'_n(f,x)|^2}{n}\Big)^{\frac{1}{2}p}dx \leqq A_p\int_0^{2\pi}|f'(x)|^p dx,$$

and

$$\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|U'_{n_k}(f,x)-\sigma'_{n_k}(f,x)|^2\Big)^{\frac{1}{2}p}dx \leqq B_p\int_0^{2\pi}|f'(x)|^p dx$$

$$(n_{k+1}/n_k > a > 1).$$

Accordingly

$$\sum_{n=1}^{N}|U'_n(f,x)-f'(x)|^2 = o(N)$$

almost everywhere and $U'_{n_k}(f,x)$ *converges to* $f'(x)$ *almost everywhere if* $n_{k+1}/n_k > a > 1$.

Proof runs after Zygmund [7]. We require three lemmas.

Lemma 3. *If* $f'(x) \varepsilon A^p$ $(1 < p < \infty)$, *then*

$$\| f - s_n \|_p \leqq C_p n^{-2} \| f'' \|_p,$$

where s_n *is the n-th partial sum of Fourier series of* $f(x)$.

For, by Marcinkiewicz [2]

$$\| f - s_n(f) \|_p \leqq n^{-1} C_p \| f' \|_p,$$

and then

$$\| f - s_n \|_p \leqq C_p n^{-1} \| f' - s'_n \|_p \leqq C_p n^{-2} \| f'' \|_p,$$

which is the required.

Lemma 4. *If* $f'(x) \varepsilon A^p (1 < p < \infty)$, *then*

$$\| U''_n(f,x) \|_p \leqq \| f'' \|_p.$$

For, by Marcinkiewicz [2] we have

$$\| U'_n(f,x) \|_p \leqq \| f' \|_p.$$

Therefore using Lemma 3, we can prove this lemma by the method analogous to Marcinkiewicz [2].

Lemma 5. *If* $p > 1$,

$$\int_0^{2\pi}\Big(\sum_{n=1}^{N}|U'_{kn}(f_n,x)|^2\Big)^{\frac{p}{2}}dx \leqq A_p\int_0^{2\pi}\Big(\sum_{n=1}^{N}|f'_n(x)|^2\Big)^{\frac{1}{2}p}dx,$$

for $f \varepsilon A^p$, *and*

$$\int_0^{2\pi}\Big(\sum_{n=1}^{N}|U''_{kn}(f_n,x)|^2\Big)^{\frac{p}{2}}dx \leqq B_p\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}|f''_n(x)|^2\Big)^{\frac{1}{2}p}dx,$$

for $f' \varepsilon A^p$.

For since, $U'_n(f,x)$ and $U''_n(f,x)$ are linear operations on $f'(x)$ and $f''(x)$ respectively, we get the lemma by using Rademacher's function. (cf. Marcinкiewicz-Zygmund (5)).

The proof of the first half of the theorem is now immediate by lemma 5, if we put $s_n(e^{\theta})$ by $U'_n(f, x)$, $s'_n(e^{i\theta})$, by $U''_n(f, x)$ and lastly $\tau_n(e^{i\theta})$ by $\sigma'_n(f, x)$ in Zygmund's argument [7].

Lemma 6. If $f(x) \varepsilon A^p$, $\sigma'_n(f, x)$ *converges to* $f'(x)$ *almost everywhere.*

This is known [2].

The later half of the theorem is immediate by using lemma 6. Thus the theorem is now completely established.

Literature

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