Non-purely non-symplectic automorphisms of odd order on $K3$ surfaces

Dedicated to Professor Shigeyuki Kondo on the occasion of his 65th birthday

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Abstract: In this paper we study non-symplectic automorphisms of odd order on $K3$ surfaces which are not purely. In particular we shall describe fixed loci of non-symplectic automorphisms of order 15 and 21.

Key words: $K3$ surface; automorphism.

1. Introduction. In this paper, we treat automorphisms of finite order on $K3$ surfaces. By the definition, a $K3$ surface has a nowhere vanishing holomorphic 2-form. An automorphism on a $K3$ surface is called *symplectic* or *non-symplectic* if it acts trivially or non-trivially on a nowhere vanishing holomorphic 2-form, respectively. Moreover an automorphism of order n of a $K3$ surface is called *purely non-symplectic* if it multiplies a nowhere vanishing holomorphic 2-form by a primitive n-th root of unity.

Symplectic automorphisms of finite order were first studied by Nikulin [11], and purely non-symplectic automorphisms have been studied by many mathematicians. Recently, studies on non-purely non-symplectic automorphisms have progressed ([1], [6], [13]). What we can say in common is that studies of fixed loci are essential as characterizations for automorphisms.

This paper is devoted to a study of non-purely non-symplectic automorphisms of odd order on $K3$ surfaces. In particular, we determine fixed loci of such automorphisms. The following is the main theorem of this paper.

Main Theorem. Let X be a $K3$ surface, ω_X a nowhere vanishing holomorphic 2-form of X and σ a non-purely non-symplectic automorphism of odd order on X.

(a) The order of σ is 15 or 21.

(b) Assume that the order of σ is 15 and σ satisfies $\sigma^*\omega_X = \zeta_3\omega_X$. Then the fixed locus is of the form $X^\sigma = \{P_{2,3}, P_{6,14}, P_{8,12}, P_{9,11}\}$.

(c) Assume that the order of σ is 15 and σ satisfies $\sigma^*\omega_X = \zeta_5\omega_X$. Then the fixed locus is of the form

$$X^\sigma = \begin{cases} \{P_{4,14}\}, \\ \{P_{1,2}, P_{4,14}, P_{4,14}, P_{5,13}, P_{7,11}, P_{8,10}\}. \end{cases}$$

(d) If the order of σ is 21 then σ satisfies $\sigma^*\omega_X = \zeta_7\omega_X$ and its fixed locus is of the form $X^\sigma = \{P_{1,2}, P_{4,20}, P_{4,20}, P_{7,17}, P_{7,17}, P_{7,17}\}$.

Here ζ_n is a primitive n-th root of unity. For the notation $P_{i,j}$, see Section 2.

Remark 1.1. As for non-purely non-symplectic automorphisms of order 21, these are essentially studied by [6, Section 7]. However they only determined the topological type of fixed loci, and did not describe their local actions for isolated fixed points. Main theorem (d) complements it.

We summarize the contents of this paper. Section 2 is a preliminary section. We recall some basic results about automorphisms on $K3$ surfaces. By simple observations, we obtain Main Theorem (a). In Section 3, we study $K3$ surfaces with a non-purely non-symplectic automorphism of order 15, and gives a proof of Main Theorem (b) and (c). In Section 4, we apply the holomorphic Lefschetz formula to determine local actions of an automorphism. This is Main Theorem (d).

2. Basic results for automorphisms on $K3$ surfaces. Let σ be an automorphism of order n on a $K3$ surface X, ζ_n a primitive n-th root of unity and (x, y) a local coordinate centered at a point in X^σ. If σ acts on the point as mapping (x, y) to $(\zeta_n^i x, \zeta_n^j y)$ then we denote it $P_{i,j}$. In this case, the action of σ for $\omega_X(= dx \wedge dy)$ is multiplication by ζ^{i+j}, hence $\sigma^*\omega_X = \zeta_n^{i+j}\omega_X$. Note that if $i \equiv 0 \mod n$

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then $P_{i,j}$ lies on a fixed curve given by $y = 0$. Thus a fixed locus of a symplectic automorphism consists of isolated fixed points, and a fixed locus of a non-symplectic automorphism generally consists of non-singular curves and isolated points.

Proposition 2.1 ([11]). *Let σ be a symplectic automorphism of order m on X. Then $m \leq 8$. Moreover, the set of fixed points of σ has cardinality* $8, 6, 4, 4, 2, 3$*, or* 2*, if* $m = 2, 3, 4, 5, 6, 7$*, or* 8*, respectively.*

As for purely non-symplectic automorphisms in this paper, it is enough for us to know the following

Proposition 2.2 ([2], [3], [14]). *Let σ be a non-symplectic automorphism of order* 3*,* 5 *or* 7*. Then the fixed locus of σ is not empty, and of the form*

$$X^\sigma = C^{(g)} \amalg \mathbf{P}^1 \amalg \cdots \amalg \mathbf{P}^1 \amalg \{P_1, \ldots, P_N\}.$$

Here $C^{(g)}$ is a genus g curve and P_k are isolated points.

Note that if the order of σ is 5 then there exist two types of fixed points hence $P_{2,4}$ and $P_{3,3}$, and if the order of σ is 7 then there exist three types of fixed points hence $P_{2,6}, P_{3,5}$ and $P_{4,4}$.

Lemma 2.3. *Let σ be a non-purely non-symplectic automorphism of finite order on a $K3$ surface. Then its fixed locus has no curves.*

Proof. Since σ is non-purely non-symplectic, we may assume that σ^m is symplectic. If there exists a fixed curve of σ then it is fixed by σ^m too. But it is a contradiction for Proposition 2.1. □

Lemma 2.4. *If a $K3$ surface has a non-purely non-symplectic automorphism of odd order then its order is* 9*,* 15*,* 21*,* 25*,* 27 *or* 33*.*

Proof. It is well-known ([9], [11]) that if a $K3$ surface has an automorphism of order I then $\Phi(I) \leq 20$ where Φ is the Euler function. Thus if I is odd then $I = 9, 15, 21, 25, 27, 33$ or prime. Clearly if I is prime then the automorphism is symplectic or purely non-symplectic. □

Proposition 2.5. *A $K3$ surface does not have a non-purely non-symplectic automorphism of order* 9*.*

Proof. It follows from [1, Theorem 0.1 (1)]. □

Remark 2.6. It does not deny that a $K3$ surface simultaneously has an symplectic automorphism of order 3 and an non-symplectic automorphism of order 3. The claim does not contradict [8, Theorem 4.1].

Proposition 2.7. *A $K3$ surface does not have non-purely non-symplectic automorphisms of order* 25*,* 27 *and* 33*.*

Proof. These follow from [9, Lemma 4.7 (4)], [9, Lemma 4.6 (2)] and [9, Lemma 4.4], respectively. □

The remaining cases are $I = 15$ and 21.

3. Order 15. Let σ be a non-purely non-symplectic automorphism of order 15. Then σ satisfies $\sigma^*\omega_X = \zeta_3\omega_X$ or $\sigma^*\omega_X = \zeta_5\omega_X$. We shall study σ in each case.

We denote by r_1, r_3, r_5 and r_{15} the rank of the eigenspace of σ^* in $H^2(X, \mathbf{C})$ relative to the eigenvalues 1, ζ_3, ζ_5 and ζ_{15} respectively. Put $S(\sigma^i) := \{x \in H^2(X, \mathbf{Z}) \mid \sigma^i(x) = x\}$.

Proposition 3.1. *The topological Euler characteristic of the fixed locus of σ is $2 + (r_1 - r_3 - r_5 + r_{15})$.*

Proof. We apply the topological Lefschetz formula:

$$\begin{aligned}
\chi(X^\sigma) &= \sum_{k=0}^{4} (-1)^k \operatorname{tr}(\sigma^* | H^k(X, \mathbf{R})) \\
&= 1 - 0 + (1 \cdot r_1 + (\zeta_3 + \zeta_3^2) r_3 \\
&\quad + (\zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4) r_5 \\
&\quad + (\zeta_{15} + \zeta_{15}^2 + \zeta_{15}^4 + \zeta_{15}^7 + \zeta_{15}^8 + \zeta_{15}^{11} \\
&\quad + \zeta_{15}^{13} + \zeta_{15}^{14}) r_{15}) - 0 + 1 \\
&= 2 + (r_1 - r_3 - r_5 + r_{15}).
\end{aligned}$$

□

3.1. The case of $\sigma^*\omega_X = \zeta_3\omega_X$. Let σ be an automorphism on X of order 15 which satisfies $\sigma^*\omega_X = \zeta_3\omega_X$. Hence the fixed locus of σ^3 consists of exactly four points, and the fixed locus of σ^5 may have some non-singular curves and some isolated points.

Proposition 3.2. *There exists one type of the fixed locus of σ:*

$$X^\sigma = \{P_{2,3}, P_{6,14}, P_{8,12}, P_{9,11}\}.$$

Proof. We see the action of σ on a fixed point $P_{i,j}$. Since σ satisfies $\sigma^*\omega_X = \zeta_{15}^{i+j}\omega_X = \zeta_3\omega_X$, we have $i + j \equiv 5 \mod 15$. Note that a fixed point of type $P_{0,5}$ lies on a fixed curve of σ. Thus X^σ consists of at most 4 isolated points of type $P_{1,4}$, $P_{2,3}$, $P_{6,14}$, $P_{7,13}$, $P_{8,12}$, $P_{9,11}$ or $P_{10,10}$.

We apply the holomorphic Lefschetz formula ([4, p. 542] and [5, p. 567]):

$$\sum_{k=0}^{2}(-1)^k \operatorname{tr}(\sigma^*|H^k(X,\mathcal{O}_X)) = \sum_{i,j}\frac{m_{i,j}}{(1-\zeta_{15}^i)(1-\zeta_{15}^j)}$$

where $m_{i,j}$ is the number of isolated fixed points of type $P_{i,j}$. Using the Serre duality $H^2(X,\mathcal{O}_X) \simeq H^0(X,\mathcal{O}_X(K_X))^\vee$, we can calculate the left-hand side as $1+\zeta_3^{-1} = 1+\zeta_{15}^{10}$. This implies

$$\begin{cases} m_{2,3} = m_{8,12} = 1 + m_{7,13}, \\ m_{6,14} = m_{9,11} = 1 + m_{1,4}, \\ m_{10,10} = 0, \end{cases}$$

hence we have $m_{2,3} = m_{6,14} = m_{8,12} = m_{9,11} = 1$ and $m_{1,4} = m_{7,13} = m_{10,10} = 0$. □

Corollary 3.3. *X^{σ^5} is of the form $C^{(5)} \amalg \mathbf{P}^1$ or $C^{(4)}$.*

Proof. Note that relations $r_1 = \operatorname{rank} S(\sigma)$, $r_1 + 2r_3 = \operatorname{rank} S(\sigma^3) = 6$, $r_1 + 4r_5 = \operatorname{rank} S(\sigma^5)$ and $r_1 + 2r_3 + 4r_5 + 8r_{15} = 22$ hold by [11, Theorem 3.1]. We remark that $r_1 > 0$ because there is an invariant ample divisor. Thus we have $(r_1, r_3, r_5, r_{15}) = (6,0,4,0)$, $(6,0,2,1)$, $(6,0,0,2)$, $(4,2,4,0)$, $(4,2,2,1)$, $(4,2,0,2)$, $(2,2,4,0)$, $(2,2,2,1)$ or $(2,2,0,2)$.

By Proposition 3.1 and Proposition 3.2, we have $(r_1, r_3, r_5, r_{15}) = (6,0,4,0)$ or $(2,2,0,2)$. Since if $(r_1, r_3, r_5, r_{15}) = (6,0,4,0)$ then σ is a purely non-symplectic automorphism of order 5, we have $\operatorname{rank} S(\sigma^5) = 2$. Thus the assertion holds by [2, Table 2] and [14, Theorem 1.1]. □

Example 3.4. Let X be the complete intersection of the following quadric and the cubic in $\mathbf{P}^5$: $\sum_{i=0}^4 X_i = \sum_{i=0}^4 X_i^2 = \sum_{i=0}^5 X_i^3 = 0$ and σ the automorphism on X satisfying $\sigma([X_0 : X_1 : X_2 : X_3 : X_4 : X_5]) = [X_1 : X_2 : X_3 : X_4 : X_0 : \zeta_3 X_5]$. It is easy to see the following

$$\begin{aligned} X^{\sigma^5} &= X \cap (X_5 = 0) \\ &= \left\{ \sum_{i=0}^4 X_i = \sum_{i=0}^4 X_i^2 = \sum_{i=0}^4 X_i^3 = 0 \right\} \\ &= C^{(4)} \end{aligned}$$

and $X^{\sigma^3} = X^\sigma = \{[1 : \zeta_5 : \zeta_5^2 : \zeta_5^3 : \zeta_5^4 : 0], [1 : \zeta_5^2 : \zeta_5^4 : \zeta_5 : \zeta_5^3 : 0], [1 : \zeta_5^3 : \zeta_5 : \zeta_5^4 : \zeta_5^2 : 0], [1 : \zeta_5^4 : \zeta_5^3 : \zeta_5^2 : \zeta_5^4 : 0]\}$.

3.2. The case of $\boldsymbol{\sigma^*\omega_X = \zeta_5\omega_X}$. Let σ be an automorphism on X of order 15 which satisfies $\sigma^*\omega_X = \zeta_5\omega_X$. Hence the fixed locus of σ^5 consists of exactly six points, and the fixed locus of σ^3 may have some non-singular curves and some isolated points.

Lemma 3.5. *The fixed locus of σ consists of one point or six points.*

Proof. Note that relations $r_1 = \operatorname{rank} S(\sigma), r_1 + 2r_3 = \operatorname{rank} S(\sigma^3), r_1 + 4r_5 = \operatorname{rank} S(\sigma^5) = 10$ and $r_1 + 2r_3 + 4r_5 + 8r_{15} = 22$ hold by [11, Theorem 3.1]. We remark that $r_1 > 0$ because there is an invariant ample divisor. Thus we have $(r_1, r_3, r_5, r_{15}) = (2,2,2,1), (2,6,2,0), (6,2,1,1), (6,6,1,0)$ or $(10,2,0,1)$, hence $\chi(X^\sigma) = 1$, -4, 6, 1 or 11, respectively by Proposition 3.1. Since X^σ does not contain a curve and consists of no more than six isolated fixed points, X^σ consists of one or six points. □

Remark 3.6. The case $(r_1, r_3, r_5, r_{15}) = (6,6,1,0)$ does not occur.

Note that if $(r_1, r_3, r_5, r_{15}) = (6,6,1,0)$ then $\operatorname{rank} S(\sigma^3) = 18$. By [3, Theorem 5.3] and [12, Main Theorem 4], a pair $(X, \langle\sigma^3\rangle)$ is isomorphic to Kondo's example [10, (7.6)] hence

$$X : y^2 = x^3 + t^3x + t^7, \ \sigma^3 : (x,y,t) \mapsto (\zeta_5^3 x, \zeta_5^2 y, \zeta_5^2 t).$$

Since the elliptic fibration $X \to \mathbf{P}^1$ has one singular fiber of type II*, one singular fiber of type III* and five singular fibers of type I$_1$, σ acts on the base $\mathbf{P}^1$ as an automorphism of order 5. Thus σ is given by $(x,y,t) \mapsto (\zeta_{15}^9 x, \zeta_{15}^6 y, \zeta_{15}^6 t)$. But it is of order 5.

Proposition 3.7. *There exist two types of fixed loci of σ:*

$$X^\sigma = \begin{cases} \{P_{4,14}\}, \\ \{P_{1,2}, P_{4,14}, P_{4,14}, P_{5,13}, P_{7,11}, P_{8,10}\}. \end{cases}$$

Proof. We see the action of σ on a fixed point $P_{i,j}$. Since σ satisfies $\sigma^*\omega_X = \zeta_{15}^{i+j}\omega_X = \zeta_5\omega_X$, we have $i + j \equiv 3 \mod 15$. Note that a fixed point of type $P_{0,3}$ lies on a fixed curve of σ. Thus X^σ consists of at most 6 isolated points of type $P_{1,2}$, $P_{4,14}$, $P_{5,13}$, $P_{6,12}$, $P_{7,11}$, $P_{8,10}$ or $P_{9,9}$.

We apply the holomorphic Lefschetz formula:

$$\sum_{k=0}^{2}(-1)^k \operatorname{tr}(\sigma^*|H^k(X,\mathcal{O}_X)) = \sum_{i,j}\frac{m_{i,j}}{(1-\zeta_{15}^i)(1-\zeta_{15}^j)}$$

where $m_{i,j}$ is the number of isolated fixed points of type $P_{i,j}$. Using the Serre duality, we can calculate the left-hand side as $1 + \zeta_5^{-1} = 1 + \zeta_{15}^{12}$. This implies

$$\begin{cases} m_{6,13} = 3 + 4m_{1,2} - 3m_{4,14} - m_{5,13}, \\ m_{7,11} = m_{1,2}, \\ m_{8,10} = m_{5,13}, \\ m_{9,9} = 1 + 3m_{1,2} - m_{4,4} - 2m_{5,13}. \end{cases}$$

Since $\sum m_{i,j} \le 6$, we have $m_{1,2} = 0$ or 1. It is easy to see if $m_{1,2} = 0$ then $(m_{4,14}, m_{5,13}) = (1,0)$ or $(0,0)$, and if $m_{1,2} = 1$ then $(m_{4,14}, m_{5,13}) = (2,1)$. But the case $(m_{1,2}, m_{4,14}, m_{5,13}) = (0,0,0)$ contradicts Lemma 3.5. Hence the assertion holds. □

Corollary 3.8. *If X^σ consists of one point then X^{σ^3} is of the form $C^{(1)} \amalg \{P_{2,4}, P_{2,4}, P_{2,4}, P_{3,3}\}$ or $\{P_{2,4}, P_{2,4}, P_{2,4}, P_{3,3}\}$.*

If X^σ consists of six points then X^{σ^3} is of the form $C^{(1)} \amalg \mathbf{P}^1 \amalg \{P_{2,4}, P_{2,4}, P_{2,4}, P_{2,4}, P_{2,4}, P_{3,3}, P_{3,3}\}$ or $\mathbf{P}^1 \amalg \{P_{2,4}, P_{2,4}, P_{2,4}, P_{2,4}, P_{2,4}, P_{3,3}, P_{3,3}\}$.

Proof. If X^σ consists of one point (or six points) then $\operatorname{rank} S(\sigma^3) = 6$ (or 10, respectively) by Lemma 3.5 and Remark 3.6. Thus the assertion holds by [3, Theorem 5.3]. □

Example 3.9. Let X' be the weighted hypersurface $X_0^5 X_1 + X_1^6 + X_2^3 + X_3^3 = 0$ in $\mathbf{P}(1,1,2,2)$ and $\bar{\sigma}$ the automorphism on X' given by $[X_0 : X_1 : X_2 : X_3] \mapsto [\zeta_5 X_0 : X_1 : \zeta_3 X_2 : \zeta_3^2 X_3]$. We remark that X' is a $K3$ surface with three singularities of type A_1 at $[0:0:1:\zeta_6], [0:0:1:-1]$ and $[0:0:1:\zeta_6^5]$ (See also [7, §10 and §13]) which are replaced by σ.

Let X be the minimal resolution of X' and σ the automorphism of X induced by $\bar{\sigma}$. The automorphism σ^3 fixes the non-singular curve of genus 1, $[1:0:0:0]$ and three points on the resolutions of the singularities. It is easy to see that σ^5 fixes six points: $[1:0:0:0], [1:\zeta_{10}^i:0:0] (i = 1,3,5,7,9)$.

Thus σ is an automorphism on X of order 15 which satisfies $\sigma^*\omega_X = \zeta_5 \omega_X$, and its fixed locus consists of exactly one point.

Example 3.10. Let X' be the weighted hypersurface $X_0^{15} + X_1^{10} + X_2^3 + X_3^2 = 0$ in $\mathbf{P}(2,3,5,15)$ and $\bar{\sigma}$ the automorphism on X' given by $[X_0 : X_1 : X_2 : X_3] \mapsto [\zeta_{15} X_0 : X_1 : \zeta_3 X_2 : X_3]$. We remark that X is a $K3$ surface with three singularities of type A_1 at $[1:0:\zeta_6:0], [1:0:-1:0]$ and $[1:0:\zeta_6^5:0]$, two singularities of type A_2 at $[0:1:0:\zeta_4]$ and $[0:1:0:-\zeta_4]$, and one singularity of type A_4 at $[0:0:1:\zeta_4]$ (See also [7, §10 and §13]).

Let X be the minimal resolution of X' and σ the automorphism of X induced by $\bar{\sigma}$. The automorphism σ^3 fixes four points on the resolutions of two singularities of type A_2, three points on the resolutions of the singularity of type A_4, a non-singular rational curve which intersects components of the resolutions of two singularities of type A_2 and the resolutions of the singularity of type A_4, and a non-singular curve of genus 1 which meets a component of the resolutions of the singularity of type A_4 (See also [3, Theorem 5.3]). Automorphisms σ and σ^5 fix six points on the resolutions of two singularities of type A_2.

Thus σ is an automorphism on X of order 15 which satisfies $\sigma^*\omega_X = \zeta_5 \omega_X$, and its fixed locus consists of exactly six points.

4. Order 21.

Let σ be a non-purely non-symplectic automorphism of order 21. Then σ satisfies $\sigma^*\omega_X = \zeta_3 \omega_X$ or $\sigma^*\omega_X = \zeta_7 \omega_X$. See also [6, Section 7.2] for details.

Proposition 4.1. *There does not exist an automorphism σ on X of order 21 which satisfies $\sigma^*\omega_X = \zeta_3 \omega_X$.*

Proof. It follows from [6, Proposition 7.11]. □

Let σ be an automorphism on X of order 21 which satisfies $\sigma^*\omega_X = \zeta_7 \omega_X$. Hence the fixed locus of σ^7 consists of exactly 6 points, and the fixed locus of σ^3 may have some non-singular curves and some isolated points.

Proposition 4.2. *There exists one type of the fixed locus of σ:*

$$X^\sigma = \{P_{1,2}, P_{4,20}, P_{4,20}, P_{7,17}, P_{7,17}, P_{7,17}\}.$$

Proof. We see the action of σ on a fixed point $P_{i,j}$. Since σ satisfies $\sigma^*\omega_X = \zeta_{21}^{i+j} \omega_X = \zeta_7 \omega_X$, we have $i + j \equiv 3 \mod 21$. Note that a fixed point of type $P_{0,7}$ lies on a fixed curve of σ. Thus X^σ consists of at most 3 isolated points of type $P_{1,2}$, $P_{4,20}$, $P_{5,19}$, $P_{6,18}$, $P_{7,17}$, $P_{8,16}$, $P_{9,15}$, $P_{10,14}$, $P_{11,13}$ or $P_{12,12}$.

We apply the holomorphic Lefschetz formula:

$$\sum_{k=0}^{2} (-1)^k \operatorname{tr}(\sigma^* | H^k(X, \mathcal{O}_X)) = \sum_{i,j} \frac{m_{i,j}}{(1-\zeta_{21}^i)(1-\zeta_{21}^j)}$$

where $m_{i,j}$ is the number of isolated fixed points of type $P_{i,j}$. Using the Serre duality, we can calculate the left-hand side as $1 + \zeta_3^{-1} = 1 + \zeta_{21}^{14}$. This implies

$$\begin{cases} m_{7,17} = -3m_{1,2} + 3m_{4,20} + m_{5,19}, \\ m_{8,16} = 1 + 4m_{1,2} - \dfrac{1}{2}(5m_{4,20} + m_{5,19} + m_{6,18}), \\ m_{9,15} = m_{13,15} - \dfrac{1}{2}(7m_{4,20} + 7m_{5,19} - m_{6,18}), \\ m_{10,14} = m_{13,15} - \dfrac{1}{2}(9m_{4,20} + m_{5,19} + 3m_{6,18}), \\ m_{11,13} = m_{13,15} - \dfrac{1}{2}(4m_{4,20} + m_{5,19} + m_{6,18}), \\ m_{12,12} = 0. \end{cases}$$

Since the fixed locus of σ consists of exactly six points by [6, Proposition 7.12], we have $m_{1,2} = 1$, $m_{4,20} = 2$, $m_{7,17} = 3$ and $m_{3,4} = m_{9,19} = m_{10,18} = m_{12,16} = m_{13,15} = m_{14,14} = 0$. $\square$

Remark 4.3. We denote by r_1, r_3, r_7 and r_{21} the rank of the eigenspace of σ^* in $H^2(X, \mathbf{C})$ relative to the eigenvalues 1, ζ_3, ζ_7 and ζ_{21} respectively. Then we have $(r_1, r_3, r_7, r_{21}) = (4, 1, 0, 1)$ and X^{σ^3} is of the form $C^{(1)} \amalg \{P_{3,5}, P_{3,5}, P_{3,5}\}$.

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