Non-purely non-symplectic automorphisms of order 6 on $K3$ surfaces

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Abstract: In this paper we study non-symplectic automorphisms of order 6 on $K3$ surfaces which are not purely. In particular we shall describe their fixed loci.

Key words: $K3$ surface; automorphism.

1. Introduction. In this paper, we treat automorphisms of finite order on $K3$ surfaces. By the definition, a $K3$ surface has a nowhere vanishing holomorphic 2-form. An automorphism on a $K3$ surface is called *symplectic* or *non-symplectic* if it acts trivially or non-trivially on a nowhere vanishing holomorphic 2-form, respectively. Moreover an automorphism of order n of a $K3$ surface is called *purely non-symplectic* if it multiplies a nowhere vanishing holomorphic 2-form by a primitive n-th root of unity.

Symplectic automorphisms of finite order were first studied by Nikulin [11], and purely non-symplectic automorphisms have been studied by many mathematicians. However, non-purely non-symplectic automorphisms have not been studied much, just [3, Proposition 2] and [1, Theorem 0.1]. What we can say in common is that studies of fixed loci are essential as characterizations for automorphisms.

This paper is devoted to a study of non-purely non-symplectic automorphisms of order 6 on $K3$ surfaces. In this case, we remark that such an automorphism acts on a nowhere vanishing holomorphic 2-form as an automorphism of order 2 or 3. The following is the main theorem of this paper.

Main Theorem. Let X be a $K3$ surface, ω_X a nowhere vanishing holomorphic 2-form of X and σ a non-purely non-symplectic automorphisms of order 6 on X. Then its fixed locus X^σ is zero-dimensional and the following holds:

(1) If σ satisfies $\sigma^*\omega_X = \zeta_3\omega_X$ then X^σ consists of 2, 5 or 8 points,

(2) If σ satisfies $\sigma^*\omega_X = -\omega_X$ then X^σ is $\emptyset$ or consists of 2, 4 or 6 points.

Here ζ_3 is a primitive 3rd root of unity.

Note that symplectic automorphisms of order 6 were studied by [11] (see also Proposition 2.1) and purely non-symplectic automorphisms of order 6 were studied by [8].

We summarize the contents of this paper. Section 2 is a preliminary section. We recall some basic results about automorphisms on $K3$ surfaces. Section 3 gives a proof of Main Theorem and examples of $K3$ surfaces with a non-purely non-symplectic automorphism of order 6. There exist different automorphisms with the same fixed locus. In order to distinguish them, we study the action of an automorphism for the 2nd cohomology of a $K3$ surface in Section 4.

The results of this paper are partially contained in the master thesis of the first-named author under the supervision of the second-named author.

2. Basic results for automorphisms on $K3$ surfaces. Let σ be an automorphism of order n on a $K3$ surface X, ζ_n a primitive n-th root of unity and (x, y) a local coordinate centered at a point in X^σ. If σ acts on the point as mapping (x, y) to $(\zeta_n^i x, \zeta_n^j y)$ then we denote it $P_{i,j}$. In this case, the action of σ for $\omega_X(= dx \wedge dy)$ is multiplication by ζ^{i+j}, hence $\sigma^*\omega_X = \zeta_n^{i+j}\omega_X$. Note that if $i \equiv 0 \mod n$ then $P_{i,j}$ lies on a fixed curve given by $y = 0$. Thus a fixed locus of a symplectic automorphism consists of isolated fixed points, and a fixed locus of a non-symplectic automorphism generally consists of non-singular curves and isolated points.

Proposition 2.1 [11]. *Let σ be a symplectic automorphism of order n on X. Then $n \le 8$. Moreover, the set of fixed points of σ has cardinality $8, 6, 4, 4, 2, 3,$ or 2, if $n = 2, 3, 4, 5, 6, 7,$ or 8, respectively.*

There are many results for non-symplectic automorphisms but we only use them in cases of

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order 2 and of order 3 in this paper, so we omit the others. See also [4, 7, 10].

Proposition 2.2 [2, 12, 13]**.**

(1) *Let σ be a non-symplectic involution. Then the fixed locus of σ is of the form*

$$X^\sigma = \begin{cases} \phi \\ C^{(1)} \amalg C^{(1)} \\ C^{(g)} \amalg \mathbf{P}^1 \amalg \cdots \amalg \mathbf{P}^1. \end{cases}$$

(2) *Let σ be a non-symplectic automorphism of order 3. Then the fixed locus of σ is of the form*

$$X^\sigma = C^{(g)} \amalg \mathbf{P}^1 \amalg \cdots \amalg \mathbf{P}^1 \amalg \{P_1, \ldots, P_n\}.$$

Here $C^{(g)}$ is a genus g curve and P_i are isolated points.

These allow us to judge whether an automorphism is symplectic or non-symplectic via its fixed locus.

Lemma 2.3. *Let σ be a non-purely non-symplectic automorphisms of order 6 on a K3 surface. Then its fixed locus has no curves.*

Proof. If there exists a fixed curve of σ then it is fixed by σ^2 and σ^3 too. But it is a contradiction for Proposition 2.1. □

3. Fixed loci. Let σ be a non-purely non-symplectic automorphisms of order 6. Then σ satisfies $\sigma^*\omega_X = \zeta_3\omega_X$ or $\sigma^*\omega_X = -\omega_X$. We shall study σ in each case.

3.1. The case of $\boldsymbol{\sigma^*\omega_X = \zeta_3\omega_X}$. Let σ be an automorphism on X of order 6 which satisfies $\sigma^*\omega_X = \zeta_3\omega_X$. Hence the fixed locus of σ^3 consists of exactly 8 points, and the fixed locus of σ^2 may have some non-singular curves and some isolated points.

Proposition 3.1. *There exists three types of fixed loci of σ:*

$$X^\sigma = \begin{cases} \{P_{3,5}, P_{3,5}\} \\ \{P_{1,1}, P_{3,5}, P_{3,5}, P_{3,5}, P_{3,5}\} \\ \{P_{1,1}, P_{1,1}, P_{3,5}, P_{3,5}, P_{3,5}, P_{3,5}, P_{3,5}, P_{3,5}\}. \end{cases}$$

Proof. We see the action of σ on a fixed point $P_{i,j}$. Since σ satisfies $\sigma^*\omega_X = \zeta_6^{i+j}\omega_X = \zeta_3\omega_X$, we have $i + j \equiv 2 \bmod 6$. Note that a fixed point of type $P_{2,6}$ lies on a fixed curve of σ. Thus X^σ consists of at most 8 isolated points of type $P_{1,1}, P_{3,5}$ or $P_{4,4}$.

We apply the holomorphic Lefschetz formula ([5, p. 542] and [6, p. 567]):

$$\sum_{k=0}^{2}(-1)^k \operatorname{tr}(\sigma^*|H^k(X, \mathcal{O}_X)) = \sum_{i,j} \frac{m_{i,j}}{(1-\zeta_6^i)(1-\zeta_6^j)}$$

where $m_{i,j}$ is the number of isolated fixed points of type $P_{i,j}$. Using the Serre duality $H^2(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(K_X))^\vee$, we can calculate the left-hand side as $1 + \zeta_3^{-1}$. This implies

$$\begin{cases} -2m_{1,1} + m_{3,5} + 6m_{4,4} = 2 \\ 4m_{1,1} - 2m_{3,5} + 4m_{4,4} = -4, \end{cases}$$

hence we have $m_{3,5} = 2m_{1,1} + 2$ and $m_{4,4} = 0$. □

Example 3.2. Let X be a quartic surface given by the homogeneous equation: $X_0X_1^3 + X_2X_3^3 + X_0^4 + X_2^4 = 0$.

(1) We consider an automorphism $\sigma : [X_0 : X_1 : X_2 : X_3] \mapsto [X_0 : \zeta_3^2 X_1 : -X_2 : \zeta_6 X_3]$ on X. It is easy to see that σ^2 is non-symplectic and σ^3 is symplectic. Moreover we have

$$\begin{aligned} X^{\sigma^3} &= X \cap (\{X_0 = X_1 = 0\} \amalg \{X_2 = X_3 = 0\}) \\ &= \{X_2X_3^3 + X_2^4 = 0\} \amalg \{X_0X_1^3 + X_0^4 = 0\} \\ &= \{[0:0:\zeta_6^i:1], [0:0:0:1]\} \\ &\quad \amalg \{[\zeta_6^i:1:0:0], [0:1:0:0]\} \ (i = 1,3,5), \\ X^{\sigma^2} &= X \cap (\{X_0 = X_2 = 0\} \amalg \{X_1 = X_3 = 0\}) \\ &= \{[0:X_1:0:X_3]\} \amalg \{X_0^4 + X_2^4 = 0\} \\ &= \mathbf{P}^1 \amalg \{[1:0:\zeta_8^j:0]\} \ (j = 1,3,5,7) \end{aligned}$$

and

$$X^\sigma = \{[0:1:0:0], [0:0:0:1]\}.$$

(2) Put $\sigma : [X_0 : X_1 : X_2 : X_3] \mapsto [X_0 : X_1 : -X_2 : \zeta_6 X_3]$. Then σ is an automorphism on X satisfying $\sigma^*\omega_X = \zeta_3\omega_X$. It is easy to check the following

$$\begin{aligned} X^{\sigma^3} &= X \cap (\{X_0 = X_1 = 0\} \amalg \{X_2 = X_3 = 0\}) \\ &= \{X_2X_3^3 + X_2^4 = 0\} \amalg \{X_0X_1^3 + X_0^4 = 0\} \\ &= \{[0:0:\zeta_6^i:1], [0:0:0:1]\} \\ &\quad \amalg \{[\zeta_6^i:1:0:0], [0:1:0:0]\} \ (i = 1,3,5), \\ X^{\sigma^2} &= \{X_0X_1^3 + X_0^4 + X_2^4 = 0\} \amalg \{[0:0:0:1]\} \\ &= C^{(3)} \amalg \{[0:0:0:1]\} \end{aligned}$$

and

$$\begin{aligned} X^\sigma &= (X \cap \{X_2 = X_3 = 0\}) \amalg \{[0:0:0:1]\} \\ &= \{X_0X_1^3 + X_0^4 = 0\} \amalg \{[0:0:0:1]\} \\ &= \{[\zeta_6^i:1:0:0], [0:1:0:0], [0:0:0:1]\} \\ &\quad (i = 1,3,5). \end{aligned}$$

Example 3.3. Let X be the weighted hypersurface $X_0^6 + X_1^6 + X_2^6 = Y^2$ in $\mathbf{P}(1,1,1,3)$ and σ the automorphism on X given by $[X_0 : X_1 : X_2 : Y] \mapsto [X_0 : X_1 : \zeta_6 X_2 : -Y]$.

Note that $[0 : 0 : \zeta_6^i : \pm 1] = [0 : 0 : 1 : \pm 1 \cdot (\zeta_6^{6-i})^3] = [0 : 0 : 1 : (\mp 1)^i]$. Thus we have

$$\begin{aligned} X^{\sigma^3} &= X \cap (\{X_2 = Y = 0\} \amalg \{X_0 = X_1 = 0\}) \\ &= \{X_0^6 + X_1^6 = 0\} \amalg \{X_2^6 = Y^2\} \\ &= \{[1 : \zeta_{12}^j : 0 : 0]\} \amalg \{[0 : 0 : 1 : \pm 1]\} \\ &\qquad (j = 1, 3, 5, 7, 9, 11), \\ X^{\sigma^2} &= X \cap (\{X_2 = 0\} \amalg \{X_0 = X_1 = 0\}) \\ &= \{X_0^6 + X_1^6 = Y^2\} \amalg \{X_2^6 = Y^2\} \\ &= \{X_0^6 + X_1^6 = Y^2\} \amalg \{[0 : 0 : 1 : \pm 1]\} \\ &= C^{(2)} \amalg \{[0 : 0 : 1 : \pm 1]\} \end{aligned}$$

and

$$X^\sigma = \{[1 : \zeta_{12}^j : 0 : 0]\} \amalg \{[0 : 0 : 1 : \pm 1]\} \qquad (j = 1, 3, 5, 7, 9, 11).$$

3.2. The case of $\sigma^* \omega_X = -\omega_X$. Let σ be an automorphism on X of order 6 which satisfies $\sigma^* \omega_X = -\omega_X$. Hence the fixed locus of σ^3 is the empty set or consists some non-singular curves, and the fixed locus of σ^2 consists of exactly 6 points.

Proposition 3.4. *There exists four types of fixed loci of* σ:

$$X^\sigma = \begin{cases} \emptyset \\ \{P_{1,2}, P_{4,5}\} \\ \{P_{1,2}, P_{1,2}, P_{4,5}, P_{4,5}\} \\ \{P_{1,2}, P_{1,2}, P_{1,2}, P_{4,5}, P_{4,5}, P_{4,5}\}. \end{cases}$$

Proof. We apply the same argument as Proposition 3.1. Since a fixed point of type $P_{3,6}$ lies on a fixed curve of σ, X^σ consists of at most 6 isolated points of type $P_{1,2}$ or $P_{4,5}$.

We apply the holomorphic Lefschetz formula:

$$\sum_{k=0}^{2} (-1)^k \operatorname{tr}(\sigma^* | H^k(X, \mathcal{O}_X)) = \sum_{i,j} \frac{m_{i,j}}{(1 - \zeta_6^i)(1 - \zeta_6^j)}.$$

Then we have $m_{1,2} = m_{4,5}$. □

Example 3.5. Let X be the weighted hypersurface $X_0^6 + X_1^6 + X_2^6 = Y^2$ in $\mathbf{P}(1, 1, 1, 3)$.

(1) Let σ be the automorphism on X given by $[X_0 : X_1 : X_2 : Y] \mapsto [X_1 : X_2 : X_0 : -Y]$. Then it is of order 6 and easy to see the following

$$\begin{aligned} X^{\sigma^3} &= X \cap \{Y = 0\} \\ &= \{X_0^6 + X_1^6 + X_2^6 = 0\} \\ &= C^{(10)} \end{aligned}$$

and

$$\begin{aligned} X^{\sigma^2} = \{&[1 : 1 : 1 : \pm\sqrt{3}], [1 : \zeta_3 : \zeta_3^2 : \pm\sqrt{3}], \\ &[1 : \zeta_3^2 : \zeta_3 : \pm\sqrt{3}]\}. \end{aligned}$$

Generally, X^σ is a subset of $X^{\sigma^2} \cap X^{\sigma^3} = \emptyset$. Thus we have $X^\sigma = \emptyset$.

(2) Let σ be the automorphism on X given by $[X_0 : X_1 : X_2 : Y] \mapsto [X_0 : \zeta_6 X_1 : \zeta_3 X_2 : Y]$. Then we have the following

$$\begin{aligned} X^{\sigma^3} &= X \cap \{X_1 = 0\} \\ &= \{Y^2 = X_0^6 + X_2^6\} \\ &= C^{(2)}, \\ X^{\sigma^2} &= \{[1 : 0 : 0 : \pm 1], [0 : 1 : 0 : \pm 1], \\ &\qquad [0 : 0 : 1 : \pm 1]\} \end{aligned}$$

and

$$X^\sigma = \{[1 : 0 : 0 : \pm 1], [0 : 0 : 1 : \pm 1]\}.$$

Example 3.6. Let X be the Fermat quartic surface given by the homogeneous equation: $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$ and σ the automorphism on X satisfying $\sigma([X_0 : X_1 : X_2 : X_3]) = [X_1 : X_2 : X_0 : -X_3]$. Then it is easy to see the following

$$\begin{aligned} X^{\sigma^3} &= X \cap (\{X_3 = 0\} \amalg \{[0 : 0 : 0 : 1]\}) \\ &= \{X_0^4 + X_1^4 + X_2^4 = 0\} \amalg \emptyset \\ &= C^{(3)}, \\ X^{\sigma^2} &= X \cap (\{X_0 = X_1 = X_2\} \amalg \{X_3 = 0\}) \\ &= \{3X_0^4 + X_3^4 = 0\} \\ &\quad \amalg \{[1 : \zeta_3 : \zeta_3^2 : 0], [1 : \zeta_3^2 : \zeta_3 : 0]\} \\ &= \{6 \text{ points}\} \end{aligned}$$

and

$$X^\sigma = \{[1 : \zeta_3 : \zeta_3^2 : 0], [1 : \zeta_3^2 : \zeta_3 : 0]\}.$$

Example 3.7. Let X' be the weighted hypersurface $X_0^6 + X_1^6 + X_2^2 X_3 + X_2 X_3^2 = 0$ in $\mathbf{P}(1, 1, 2, 2)$ and $\bar{\sigma}$ be the automorphism on X' given by $[X_0 : X_1 : X_2 : X_3] \mapsto [\zeta_6 X_0 : \zeta_3 X_1 : X_2 : X_3]$. We remark that X' is a $K3$ surface with three singularities of type A_1 at $[0 : 0 : 1 : 0], [0 : 0 : 1 : -1], [0 : 0 : 0 : 1]$ (See also [9, §10 and §13]), and these are fixed points of $\bar{\sigma}$.

Let X be the minimal resolution of X' and σ the automorphism of X induced by $\bar{\sigma}$. Since each singular point which is fixed by $\bar{\sigma}$ induces 2 fixed points of σ, X^σ consists of exactly 6 points.

Moreover we see fixed locus of σ^3. We remark that

$$\begin{aligned}(X')^{\bar{\sigma}^3} &= (X' \cap \{X_0 = 0\}) \cup (X' \cap \{X_1 = 0\}) \\ &= \{X_1^6 + X_2^2 X_3 + X_2 X_3^2 = 0\} \\ &\quad \cup \{X_0^6 + X_2^2 X_3 + X_2 X_3^2 = 0\}\end{aligned}$$

and $\mathbf{P}(1,2,2) \simeq \mathbf{P}^2$, hence two cubic curves intersect at $[0:0:1:0], [0:0:1:-1], [0:0:0:1]$. Thus, after blowing-up, we can see that X^{σ^3} consists of two non-singular curves of genus 1.

Recall that non-symplectic automorphisms of order 3 are determined by only fixed loci [2, 13]. Example 3.2 (1) and the following Example 3.8 give automorphisms of order 6 which fix exactly 2 points. But fixed loci of their squares (automorphisms of order 3) are different. This implies that Proposition 3.1 and Proposition 3.4 do not give the classification of automorphisms.

Example 3.8. Let X be the complete intersection of the following quadric and the cubic in $\mathbf{P}^4$: $X_0^2 + X_1^2 + X_2^2 + X_3^2 = X_0^3 + X_1^3 + X_0 X_2^2 + X_1 X_3^2 + X_4^3 = 0$ and σ the automorphism on X satisfying $\sigma([X_0 : X_1 : X_2 : X_3 : X_4]) = [X_0 : X_1 : -X_2 : -X_3 : \zeta_3 X_4]$. It is easy to see the following

$$\begin{aligned}X^{\sigma^3} &= X \cap (\{X_2 = X_3 = 0\} \amalg \{X_0 = X_1 = X_4 = 0\}) \\ &= \{X_0^2 + X_1^2 = X_0^3 + X_1^3 + X_4^3 = 0\} \\ &\quad \amalg \{X_2^2 + X_3^2 = 0\} \\ &= \{8 \text{ points}\}, \\ X^{\sigma^2} &= X \cap (X_4 = 0) \\ &= \left\{\sum_{i=0}^{3} X_i^2 = X_0^3 + X_1^3 + X_0 X_2^2 + X_1 X_3^2 = 0\right\} \\ &= C^{(4)}\end{aligned}$$

and

$$\begin{aligned}X^{\sigma} &= X \cap \left(\left\{\sum_{i=2}^{4} X_i = 0\right\} \amalg \{X_0 = X_1 = X_4 = 0\}\right) \\ &= \emptyset \amalg \{X_2^2 + X_3^2 = 0\} \\ &= \{[0:0:1:\pm\sqrt{-1}:0]\}.\end{aligned}$$

4. Actions for the 2nd cohomology.

We denote by r_1, r_2, r_3 and r_6 the rank of the eigenspace of σ^* in $H^2(X, \mathbf{C})$ relative to the eigenvalues 1, -1, ζ_3 and ζ_6 respectively. Put $S(\sigma^i) := \{x \in H^2(X, \mathbf{Z}) \mid \sigma^i(x) = x\}$. Then relations $r_1 = \operatorname{rank} S(\sigma)$, $r_1 + r_2 = \operatorname{rank} S(\sigma^2)$, $r_1 + 2r_3 = \operatorname{rank} S(\sigma^3)$ and $r_1 + r_2 + 2r_3 + 2r_6 = 22$ hold by [11, Theorem 3.1]. We remark that $r_1 > 0$ because there is an invariant ample divisor.

Lemma 4.1. *Let $\chi(X^\sigma)$ be the Euler characteristic of X^σ. Then we have $\chi(X^\sigma) = -20 + 2r_1 + r_3 + 3r_6$.*

Proof. We apply the topological Lefschetz formula:

$$\begin{aligned}\chi(X^\sigma) &= \sum_{k=0}^{4} (-1)^k \operatorname{tr}(\sigma^* | H^k(X, \mathbf{R})) \\ &= 1 - 0 + (1 \cdot r_1 + (-1) \cdot r_2 + (\zeta_3 + \zeta_3^2) r_3 \\ &\quad + (\zeta_6 + \zeta_6^5) r_6) - 0 + 1 \\ &= 2 + (r_1 - r_2 - r_3 + r_6) \\ &= -20 + 2r_1 + r_3 + 3r_6.\end{aligned}$$

□

4.1. The case of $\sigma^*\omega_X = \zeta_3\omega_X$. In this case, σ^2 is a non-symplectic automorphism of order 3 and σ^3 is a symplectic involution. Since $\operatorname{rank} S(\sigma^3) (= r_1 + 2r_3) = 14$ by [11, §10], pairs (r_1, r_3) are $(2,6)$, $(4,5)$, $(6,4)$, $(8,3)$, $(10,2)$, $(12,1)$ or $(14,0)$.

Lemma 4.2. *The case $(r_1, r_3) = (14, 0)$ does not occur.*

Proof. If $(r_1, r_3) = (14, 0)$ then $\chi(X^\sigma) = 8 + 3r_6$ by Lemma 4.1. Since the order of σ is 6, $r_6 \neq 0$. It contradicts Proposition 3.1. □

Proposition 4.3. *The following hold*:

(1) *If X^σ consists of 2 points then $(r_1, r_2, r_3, r_6) = (2,0,6,4)$, $(4,2,5,3)$, $(6,4,4,2)$, $(8,6,3,1)$ or $(10,8,2,0)$.*

(2) *If X^σ consists of 5 points then $(r_1, r_2, r_3, r_6) = (4,0,5,4)$, $(6,2,4,3)$, $(8,4,3,2)$, $(10,6,2,1)$ or $(12,8,1,0)$.*

(3) *If X^σ consists of 8 points then $(r_1, r_2, r_3, r_6) = (6,0,4,4)$, $(8,2,3,3)$, $(10,4,2,2)$ or $(12,6,1,1)$.*

Proof. Since $r_2 + 2r_6 = 8$ and $\chi(X^\sigma) = -13 + 3r_1/2 + 3r_6$ by $r_1 + 2r_3 = 14$ and Lemma 4.1, it is easy to see all possibilities of pairs (r_1, r_2, r_3, r_6) in each case. But if $(\chi(X^\sigma), r_1, r_3) = (2, 12, 1)$ then we have $r_6 < 0$, and if $(\chi(X^\sigma), r_1, r_3) = (5, 2, 6)$ or $(8, 2, 6)$ then we have $r_2 < 0$. These are contradictions. □

The number of isolated fixed points of σ^2 is $(r_1 + r_2 - 2)/2$ by [2, Theorem 2.2] and [13, Theorem 1.1]. Thus Example 3.2 (1) is of type $(r_1, r_2, r_3, r_6) = (6,4,4,2)$ and Example 3.8 is of type $(r_1, r_2, r_3, r_6) = (2,0,6,4)$.

By the same argument, it is easy to see that Example 3.2 (2) is of type $(r_1, r_2, r_3, r_6) = (4,0,5,4)$ and Example 3.3 is of type $(r_1, r_2, r_3, r_6) = (6,0,4,4)$.

4.2. The case of $\sigma^*\omega_X = -\omega_X$. In this case,

σ^2 is a symplectic automorphism of order 3 and σ^3 is a non-symplectic involution. Note that $\operatorname{rank} S(\sigma^2)(= r_1 + r_2) = 10$ by [11, §10], hence $r_3 + r_6 = 6$ and $\chi(X^\sigma) = -14 + 2r_1 + 2r_6$. By using these equations, we have the following

Proposition 4.4. *The following hold*:

(1) *If X^σ is empty then* $(r_1, r_2, r_3, r_6) = (1, 9, 0, 6)$, $(2, 8, 1, 5)$, $(3, 7, 2, 4)$, $(4, 6, 3, 3)$, $(5, 5, 4, 2)$, $(6, 4, 5, 1)$ *or* $(7, 3, 6, 0)$.

(2) *If X^σ consists of* 2 *points then* $(r_1, r_2, r_3, r_6) = (2, 8, 0, 6)$, $(3, 7, 1, 5)$, $(4, 6, 2, 4)$, $(5, 5, 3, 3)$, $(6, 4, 4, 2)$, $(7, 3, 5, 1)$ *or* $(8, 2, 6, 0)$.

(3) *If X^σ consists of* 4 *points then* $(r_1, r_2, r_3, r_6) = (3, 7, 0, 6)$, $(4, 6, 1, 5)$, $(5, 5, 2, 4)$, $(6, 4, 3, 3)$, $(7, 3, 4, 2)$, $(8, 2, 5, 1)$ *or* $(9, 1, 6, 0)$.

(4) *If X^σ consists of* 6 *points then* $(r_1, r_2, r_3, r_6) = (4, 6, 0, 6)$, $(5, 5, 1, 5)$, $(6, 4, 2, 4)$, $(7, 3, 3, 3)$, $(8, 2, 4, 2)$ *or* $(9, 1, 5, 1)$.

Proof. If $X^\sigma = \emptyset$ then we have $1 \leq r_1 \leq 7$, hence $(r_1, r_2, r_3, r_6) = (1, 9, 0, 6)$, $(2, 8, 1, 5)$, $(3, 7, 2, 4)$, $(4, 6, 3, 3)$, $(5, 5, 4, 2)$, $(6, 4, 5, 1)$ or $(7, 3, 6, 0)$.

If $\chi(X^\sigma) = 2$ then we have $2 \leq r_1 \leq 8$, hence $(r_1, r_2, r_3, r_6) = (2, 8, 0, 6)$, $(3, 7, 1, 5)$, $(4, 6, 2, 4)$, $(5, 5, 3, 3)$, $(6, 4, 4, 2)$, $(7, 3, 5, 1)$ or $(8, 2, 6, 0)$.

If $\chi(X^\sigma) = 4$ then we have $3 \leq r_1 \leq 9$, hence $(r_1, r_2, r_3, r_6) = (3, 7, 0, 6)$, $(4, 6, 1, 5)$, $(5, 5, 2, 4)$, $(6, 4, 3, 3)$, $(7, 3, 4, 2)$, $(8, 2, 5, 1)$ or $(9, 1, 6, 0)$.

If $\chi(X^\sigma) = 6$ then we have $4 \leq r_1 \leq 10$, hence $(r_1, r_2, r_3, r_6) = (4, 6, 0, 6)$, $(5, 5, 1, 5)$, $(6, 4, 2, 4)$, $(7, 3, 3, 3)$, $(8, 2, 4, 2)$, $(9, 1, 5, 1)$ or $(10, 0, 6, 0)$. Since the order of σ is 6, the case $(r_1, r_2, r_3, r_6) = (10, 0, 6, 0)$ does not occur. □

We remark that $r_1 + 2r_3$ is the rank of $S(\sigma^3)$. If X^{σ^3} consists of exactly one non-singular curve of genus 10 then we have $r_1 + 2r_3 = 1$ by the classification of non-symplectic involutions [12]. Thus Example 3.5 (1) is of type $(r_1, r_2, r_3, r_6) = (1, 9, 0, 6)$. By the same argument, we can check that Example 3.5 (2) is of type $(r_1, r_2, r_3, r_6) = (5, 5, 2, 4)$, Example 3.6 is of type $(r_1, r_2, r_3, r_6) = (4, 6, 2, 4)$ and Example 3.7 is of type $(r_1, r_2, r_3, r_6) = (6, 4, 2, 4)$.

Remark 4.5. It is expected that there exists an example for each remaining (r_1, r_2, r_3, r_6).

Example 4.6. Let X be the complete intersection of the quadric and the cubic in $\mathbf{P}^4$: $X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_3 X_4^2 = 0$ and σ the automorphism on X satisfying $\sigma([X_0 : X_1 : X_2 : X_3 : X_4]) = [X_1 : X_2 : X_0 : X_3 : -X_4]$. It is easy to see the following

$$\begin{aligned} X^{\sigma^3} &= X \cap \{X_4 = 0\} \\ &= \{X_0^2 + X_1^2 + X_2^2 + X_3^2 \\ &\qquad = X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0\} \\ &= C^{(1)}, \\ X^{\sigma^2} &= X \cap \{X_0 = X_1 = X_2\} \\ &= \{3Y^2 + X_3^2 + X_4^2 = 3Y^3 + X_3^3 + X_3 X_4^2 = 0\} \\ &= \{6 \text{ points}\} \quad (\text{by the Bezout theorem}) \end{aligned}$$

and

$$X^\sigma = \emptyset.$$

Since X^{σ^3} consists of exactly one non-singular curve of genus 1, we have $r_1 + 2r_3 = 1$. This is of type $(r_1, r_2, r_3, r_6) = (4, 6, 3, 3)$.

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