Bochner-Schoenberg-Eberlein property for abstract Segal algebras

By Zeinab Kamali and Mahmood Lashkarizadeh Bami

Department of Mathematics, Faculty of Science, University of Isfahan, Isfahan, Iran

(Communicated by Masaki Kashiwara, m.j.a., Oct. 15, 2013)

Abstract: Let $\mathcal{A}$ be a BSE Banach algebra and $\mathcal{B}$ be an essential abstract Segal algebra with respect to $\mathcal{A}$. In this paper we present a necessary and sufficient condition for $\mathcal{B}$ to be a BSE algebra as well. Furthermore we study BSE property of some certain abstract Segal algebras which are not discussed in previous works.

Key words: Abstract Segal algebra; BSE algebra; Δ-weak bounded approximate identity.

1. Introduction. Let $\mathcal{A}$ be a commutative Banach algebra without order. Denote by $\Delta(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ the Gelfand spectrum and the multiplier algebra of $\mathcal{A}$, respectively. A bounded continuous function σ on $\Delta(\mathcal{A})$ is called a *BSE-function* if there exists a constant $C > 0$ such that for every finite number of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and the same number of complex numbers $c_1, \ldots, c_n$, the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C.\left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds. The BSE-norm of σ, $\|\sigma\|_{BSE}$, is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(\mathcal{A}))$. Takahasi and Hatori [19] showed that under the norm $\|.\|_{BSE}$, $C_{BSE}(\Delta(\mathcal{A}))$ is a commutative semisimple Banach algebra. The algebra $\mathcal{A}$ is called a *BSE-algebra* (or said to have the *BSE-property*) if the BSE-functions on $\Delta(\mathcal{A})$ are precisely the Gelfand transforms of the elements of $\mathcal{M}(\mathcal{A})$. That is $\mathcal{A}$ is a BSE-algebra if and only if

$$C_{BSE}(\Delta(\mathcal{A})) = \widehat{\mathcal{M}(\mathcal{A})}.$$

The abbreviation BSE stands for Bochner-Schoenberg-Eberlein and refers to the famous theorem, proved by Bochner and Schoenberg [2,18] for the additive group of real numbers and in general by Eberlein [6] for locally compact abelian groups G, saying that, in the above terminology, the group algebra $L^1(G)$ is a BSE-algebra (See [17] for a proof).

The notion of BSE-algebra and the algebra of BSE-functions were introduced and studied by Takahasi and Hatori [19,20] and later by Kaniuth and Ülger [12]. Also the authors have got some new results on BSE algebras such as BSE property of direct sum of Banach algebras [11].

In 2000, Inoue and Takahasi [8] proved that every Segal algebra $S(G)$ of a locally compact group G is a BSE-algebra if and only if it has a Δ-weak bounded approximate identity.

In this paper we generalize this result to abstract Segal algebras. Indeed, we prove that an abstract essential Segal algebra with respect to a BSE-algebra is BSE if and only if it has a Δ-weak bounded approximate identity.

In last section, we study the BSE property for certain abstract Segal algebras which are not discussed before.

2. Preliminaries.

Definition 2.1. Let $(\mathcal{A}, \|.\|_{\mathcal{A}})$ be a Banach algebra. A Banach algebra $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is an abstract Segal algebra with respect to $\mathcal{A}$ if

(i) $\mathcal{B}$ is a dense ideal in $\mathcal{A}$.

(ii) There exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$, for all $b \in \mathcal{B}$.

(iii) There exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$, for all $a, b \in \mathcal{B}$.

We quote the following result from [3]:

Proposition 2.2. *Let $(\mathcal{B}, \|.\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to the commutative Banach algebra $(\mathcal{A}, \|.\|_{\mathcal{A}})$. Then $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ are homeomorphic.*

Definition 2.3. An ideal $\mathcal{B}$ in a Banach algebra $\mathcal{A}$ is called essential, if

2010 Mathematics Subject Classification. Primary 46Jxx; Secondary 22D15.

doi: 10.3792/pjaa.89.107

$$\mathcal{B} = \{ax : a \in \mathcal{A}, x \in \mathcal{B}\}.$$

Dunford [4] proved that any Segal algebra $\mathcal{S}(G)$ on a locally compact group G is an essential ideal in $L^1(G)$.

A linear bounded operator on $\mathcal{A}$ is called a *multiplier* if it satisfies $xT(y) = T(xy)$ for all $x, y \in \mathcal{A}$. The set $\mathcal{M}(\mathcal{A})$ of all multipliers on $\mathcal{A}$ is a unital commutative Banach algebra, called the *multiplier algebra* of $\mathcal{A}$.

For each $T \in \mathcal{M}(\mathcal{A})$ there exists a unique continuous function $\widehat{T}$ on $\Delta(\mathcal{A})$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi)$ for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. See [15] for a proof.

A bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ is called a bounded approximate identity for $\mathcal{A}$ if it satisfies $\|e_\alpha a - a\| \to 0$ for all $a \in \mathcal{A}$. A bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ is called a Δ-weak bounded approximate identity for $\mathcal{A}$ if it satisfies $\varphi(e_\alpha a) \to \varphi(a)$ $(a \in \mathcal{A},\ \varphi \in \Delta(\mathcal{A})$. Such approximate identities were studied in [10]. Takahasi and Hatori obtained the following result in [19]:

Proposition 2.4. *Let $\mathcal{A}$ be a commutative Banach algebra without order. $\mathcal{A}$ has a Δ-weak bounded approximate identity if and only if $\widehat{\mathcal{M}(\mathcal{A})} \subseteq C_{BSE}(\Delta(\mathcal{A}))$.*

3. Main result.

Theorem 3.1. *Let $(\mathcal{A}, \|.\|_\mathcal{A})$ be a BSE-algebra and $(\mathcal{B}, \|.\|_\mathcal{B})$ an essential abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{B}$ is a BSE-algebra if and only if it has a Δ-weak bounded approximate identity.*

Proof. Suppose that $\mathcal{B}$ is a BSE-algebra. Then by Proposition 2.4 it has a Δ-weak bounded approximate identity. Conversely, suppose that $\mathcal{B}$ has a Δ-weak bounded approximate identity, then by the same proposition,

$$\widehat{\mathcal{M}(\mathcal{B})} \subseteq C_{BSE}(\Delta(\mathcal{B})).$$

So it remains to show that $C_{BSE}(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{M}(\mathcal{B})}$. Suppose that $\sigma \in C_{BSE}(\Delta(\mathcal{B}))$. Then there exists a positive number C such that for any finite number of $\varphi_1, \ldots, \varphi_n \in \Delta(\mathcal{B})$ and $c_1, \ldots, c_n \in \mathbf{C}$,

$$\left|\sum_{i=1}^{n} c_i\sigma(\varphi_i)\right| \le C \left\|\sum_{i=1}^{n} c_i\varphi_i\right\|_{\mathcal{B}^*}.$$

Now for every $f \in \mathcal{A}^*$ and $x \in \mathcal{A}$, we have $|f(x)| \le \|f\|_{\mathcal{A}^*}\|x\|_\mathcal{A}$. By definition of abstract Segal algebra, there exists $M > 0$ such that $\|x\|_\mathcal{A} \le M\|x\|_\mathcal{B}$ $(x \in \mathcal{B})$. It follows that

$$|f(x)| \le \|f\|_{\mathcal{A}^*}\|x\|_\mathcal{A} \le M\|f\|_{\mathcal{A}^*}\|x\|_\mathcal{B} \quad (x \in \mathcal{B}).$$

Hence $\|f\|_{\mathcal{B}^*} \le M\|f\|_{\mathcal{A}^*}$. Especially, we have:

$$\left|\sum_{i=1}^{n} c_i\sigma(\varphi_i)\right| \le C \left\|\sum_{i=1}^{n} c_i\varphi_i\right\|_{\mathcal{B}^*} \le CM \left\|\sum_{i=1}^{n} c_i\varphi_i\right\|_{\mathcal{A}^*}.$$

By Proposition 2.2, $\Delta(\mathcal{B})$ is homeomorphic to $\Delta(\mathcal{A})$ and we may consider $\varphi_1, \ldots, \varphi_n \in \Delta(\mathcal{A})$. It means that $\sigma \in C_{BSE}(\Delta(\mathcal{A}))$. Since A is a BSE-algebra, $\sigma \in \widehat{\mathcal{M}(A)}$. Therefore there exists $T \in \mathcal{M}(\mathcal{A})$ such that $\sigma = \widehat{T}$. We have to show that $T|_\mathcal{B} \in \mathcal{M}(\mathcal{B})$. Since $T \in \mathcal{M}(\mathcal{A})$, it is obvious that $T(xy) = T(x)y$ $(x, y \in \mathcal{B})$. So it is enough to show that $T\mathcal{B} \subseteq \mathcal{B}$. Indeed, if it is shown that $T\mathcal{B} \subseteq \mathcal{B}$, then T is continuous in the $\|.\|_\mathcal{B}$-topology by the closed graph theorem because $\mathcal{B}$ has no nonzero annihilators. Let $x \in \mathcal{B}$. Since $\mathcal{B}$ is an essential ideal of $\mathcal{A}$, there exist $a \in \mathcal{A}$ and $y \in \mathcal{B}$ such that $x = ay$ and hence

$$T(x) = T(ay) = T(a)y \in \mathcal{B}.$$

Thus $\sigma \in \widehat{\mathcal{M}(\mathcal{B})}$. Hence $\mathcal{B}$ is a BSE-algebra. □

Remark 3.2. When $\mathcal{B}$ is an abstract Segal algebra with respect to $\mathcal{A}$, as it is shown in the proof of Theorem 3.1, we have

$$C_{BSE}(\Delta(\mathcal{B})) \subseteq C_{BSE}(\Delta(\mathcal{A})).$$

4. BSE property of certain abstract Segal algebras.

In this section we study the BSE property of some abstract Segal algebras which are not discussed in [8].

4.1. Segal algebras of compact abelian groups.

A dense ideal $S(G)$ of the convolution group algebra $L^1(G)$ of a locally compact group G is said to be a Segal algebra if it satisfies the following conditions:

(a) $S(G)$ is a Banach space under some norm $\|.\|_S$ and $\|f\|_S \ge \|f\|_1$.

(b) $S(G)$ is left translation invariant, i.e. $\|L_x f\|_S = \|f\|_S$ for all $x \in G$ and $f \in S(G)$, and the map $x \mapsto L_x f$ from G into $S(G)$ is continuous.

Every Segal algebra is an abstract segal algebra with respect to $L^1(G)$ (see [9], Proposition 1).

Proposition 4.1. *Let G be an abelian compact group. Then a Segal algebra $S(G)$ is a BSE algebra if and only if $S(G) = L^1(G)$.*

Proof. The "if" part is clear, since $L^1(G)$ is a BSE algebra. Conversely, suppose that $S(G)$ is a BSE algebra. Since G is an abelian compact group, then $S(G)$ is an ideal in its second dual [16]. By semisimplicity of $S(G)$ and by Theorem 3.1 of [12], it has a bounded approximate identity which by Theorem 1.2 of [3] implies that $S(G) = L^1(G)$. $\square$

For a locally compact group G, let $A(G)$ be the Fourier algebra defined in [5] and let

$$\mathfrak{L}A(G) = A(G) \cap L^1(G)$$

with norm

$$\|f\| = \|f\|_{A(G)} + \|f\|_1.$$

$(\mathfrak{L}A(G), \|.\|)$ with convolution product is a Segal algebra, called Lebesgue-Fourier algebra. Note that $\mathfrak{L}A(G)$ with pointwise multiplication is an abstract Segal algebra of $A(G)$.

The concept of Lebesgue-Fourier algebra was introduced and extensively studied by Ghahramani and Lau [7].

Corollary 4.2. *Let G be an abelian compact group. Then the Banach algebra $\mathfrak{L}A(G)$ is a BSE algebra if and only if G is finite.*

Proof. By Proposition 2.3 of [7], $\mathfrak{L}A(G) = L^1(G)$ if and only if G is discrete. Then by Proposition 4.1, $\mathfrak{L}A(G)$ is BSE if and only if G is discrete and by compactness of G, if and only if it is finite. $\square$

Remark 4.3. Let G be a discrete group and suppose that $\mathfrak{L}A(G)$ is equipped with the pointwise product. Then $\mathfrak{L}A(G)$ is a BSE algebra if and only if G is finite. In fact, when G is discrete, $\mathfrak{L}A(G) = l^1(G)$ with poinwise multiplication and this algebra is BSE if and only if G is finite [20].

4.2. $\mathcal{W}^p$-algebras. Consider the additive group of vectors in $\mathbf{R}^n$ and

$$Q = \left\{ x = (x_1, \ldots, x_n) \in \mathbf{R}^n : -\frac{1}{2} \leq x_i < \frac{1}{2} \, (1 \leq i \leq n) \right\}.$$

For $t \in \mathbf{R}^n$, define $Q_t := \{t + x : x \in Q\}$ and f_t denotes the translated function $f_t(x) = f(x - t)$. For an arbitrary set A, χ_A will denote the characteristic function of A. For simplicity we write χ_t instead of χ_{Q_t}.

For $1 < p < \infty$, let

$$\mathcal{W}^p = \left\{ f \in L^1(\mathbf{R}^n) : \sum_{m \in \mathbf{Z}^n} \|\chi_m f\|_p < \infty \right\}.$$

By Proposition 3.1 of [14], $\mathcal{W}^p$ is a Segal algebra with respect to $L^1(\mathbf{R}^n)$ by the norm

$$\|f\|_{\mathcal{W}^p} = \max_{t \in Q} \sum_{m \in \mathbf{Z}^n} \|\chi_m f_t\|_p \quad (f \in \mathcal{W}^p).$$

Proposition 4.4. *The Segal algebra $\mathcal{W}^p$ is not BSE.*

Proof. By Corollary 3.8 of [14], there is a multiplier in $M(\mathcal{W}^p)$ which is not a measure. It means that there exists $T \in M(\mathcal{W}^p)$ and $T \notin M(\mathbf{R}^n)$. Since $M(\mathbf{R}^n)$ is a semisimple Banach algebra, $\widehat{T} \in \widehat{M(\mathcal{W}^p)}$ and $\widehat{T} \notin \widehat{M(\mathbf{R}^n)}$. $L^1(\mathbf{R}^n)$ is a BSE algebra, then $\widehat{T} \notin C_{BSE}(\Delta(L^1(\mathbf{R}^n))$ and consequently by Remark 3.2, $\widehat{T} \notin C_{BSE}(\Delta(\mathcal{W}^p)))$. It follows that $\widehat{M(\mathcal{W}^p)} \neq C_{BSE}(\Delta(\mathcal{W}^p))$ and $\mathcal{W}^p$ is not a BSE algebra. $\square$

Corollary 4.5. *$\mathcal{W}^p$ has no Δ-weak bounded approximate identity.*

Proof. By Proposition 4.4 and Theorem 3.1, the result is obvious. $\square$

4.3. C^*-Segal algebra $C_0^w(X)$. Let X be a locally compact Hausdorff space, and let $w : X \to \mathbf{R}$ be an upper semicontinuous function such that $w(t) \geq 1$ for every $t \in X$. Define

$$C_0^w(X) := \{f \in C(X) : fw \text{ vanishes at infinity on } X\},$$

where $C(X)$ denotes the set of all continuous complex-valued functions on X.

Equipped with pointwise operations and the weighted supremum norm

$$\|f\|_w := \sup_{t \in X} w(t)|f(t)| \ (f \in C_0^w(X)),$$

$C_0^w(X)$ is a self-adjoint C*-Segal algebra (abstract segal algebra with respect to $C_0(X)$) [13].

By Proposition 2.2, $\Delta(C_0^w(X)) = \Delta(C_0(X)) = X$. In fact, the function $x \leftrightarrow \phi_x$, where $\phi_x(f) = f(x)(x \in X, f \in C_0^w(X))$, is a homeomorphism from X onto $\Delta(C_0^w(X))$.

Proposition 4.6. *$C_0^w(X)$ is a BSE algebra if and only if w is bounded.*

Proof. When w is bounded, by [1], $C_0^w(X) = C_0(X)$ is a C^*-algebra and so by Theorem 3 of [19] is a BSE algebra. Now suppose that $C_0^w(X)$ is a BSE algebra. Then by Proposition 2.4, it has a Δ-weak approximate identity. It means that there exists a bounded net $\{f_\alpha\}_\alpha \in C_0^w(X)$ such that $\lim_\alpha f_\alpha(t) = 1$, for all $t \in X$ and there exists $\beta > 0$ such that

$$\sup_\alpha \|f_\alpha\|_w = \sup_\alpha \sup_{t \in X} |f_\alpha(t)|w(t) \leq \beta.$$

On the other hand, $\lim_\alpha f_\alpha(t) = 1$ implies that $\lim_\alpha |f_\alpha(t)|w(t) = w(t)$. Thus $w(t) \leq \beta$ $(t \in X)$ which means that w is bounded. $\square$

As it is shown in Corollary 3.6 of [1], $C_0^w(X)$ has a bounded approximate identity if and only if w is bounded. By Theorem 3.1 and Proposition 4.6, we conclude the following result:

Corollary 4.7. *$C_0^w(X)$ has a Δ-weak bounded approximate identity if and only if w is bounded.*

References

[1] J. Arhippainen and J. Kauppi, Generalization of the B^*-algebra $(C_0(X), \| \|_\infty)$, Math. Nachr. **282** (2009), no. 1, 7–15.

[2] S. Bochner, A theorem on Fourier-Stieltjes integrals, Bull. Amer. Math. Soc. **40** (1934), no. 4, 271–276.

[3] J. T. Burnham, Closed ideals in subalgebras of Banach algebras. I, Proc. Amer. Math. Soc. **32** (1972), 551–555.

[4] D. H. Dunford, Segal algebras and left normed ideals, J. London Math. Soc. (2) **8** (1974), 514–516.

[5] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France **92** (1964), 181–236.

[6] W. F. Eberlein, Characterizations of Fourier-Stieltjes transforms, Duke Math. J. **22** (1955), 465–468.

[7] F. Ghahramani and A. T. M. Lau, Weak amenability of certain classes of Banach algebras without bounded approximate identities, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 2, 357–371.

[8] J. Inoue and S.-E. Takahasi, Constructions of bounded weak approximate identities for Segal algebras on LCA groups, Acta Sci. Math. (Szeged) **66** (2000), no. 1–2, 257–271.

[9] B. E. Johnson, Weak amenability of group algebras, Bull. London Math. Soc. **23** (1991), no. 3, 281–284.

[10] C. A. Jones and C. D. Lahr, Weak and norm approximate identities are different, Pacific J. Math. **72** (1977), no. 1, 99–104.

[11] Z. Kamali, M. Lashkarizadeh Bami, The multiplier algebra and BSE property of direct sum of Banach Algebras, Bull. Aust. Math. Soc. **88** (2013), no. 2, 250–258.

[12] E. Kaniuth and A. Ülger, The Bochner-Schoenberg-Eberlein property for commutative Banach algebras, especially Fourier and Fourier-Stieltjes algebras, Trans. Amer. Math. Soc. **362** (2010), no. 8, 4331–4356.

[13] J. Kauppi and M. Mathieu, C^*-Segal algebras with order unit, J. Math. Anal. Appl. **398** (2013), no. 2, 785–797.

[14] H. E. Krogstad, Multipliers of Segal algebras, Math. Scand. **38** (1976), no. 2, 285–303.

[15] R. Larsen, *An introduction to the theory of multipliers*, Springer, New York, 1971.

[16] H. S. Mustafayev, Segal algebra as an ideal in its second dual space, Turkish J. Math. **23** (1999), no. 2, 323–332.

[17] W. Rudin, *Fourier analysis on groups*, Wiley Interscience, New York, 1984.

[18] I. J. Schoenberg, A remark on the preceding note by Bochner, Bull. Amer. Math. Soc. **40** (1934), no. 4, 277–278.

[19] S. Takahasi and O. Hatori, Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein type-theorem, Proc. Amer. Math. Soc. **110** (1990), no. 1, 149–158.

[20] S. Takahasi and O. Hatori, Commutative Banach algebras and BSE-inequalities, Math. Japon. **37** (1992), no. 4, 607–614.