Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$

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(Communicated by Shigefumi MORI, M.J.A., Feb. 12, 2013)

Abstract: In this paper, we find all Fibonacci and Lucas numbers written in the form $2^a + 3^b + 5^c$, in nonnegative integers a, b, c, with $\max\{a, b\} \le c$.

Key words: Fibonacci; Lucas; linear forms in logarithms; reduction method.

1. Introduction. Let $(F_n)_{n\ge 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \ge 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [5] together with its very extensive annotated bibliography for additional references and history). We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\ge 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

The problem of finding for Fibonacci and Lucas numbers of a particular form has a very rich history. Maybe the most outstanding result on this subject is due to Bugeaud, Mignotte and Siksek [1, Theorem 1] who showed that 0, 1, 8, 144 and 1, 4 are the only Fibonacci and Lucas numbers, respectively, of the form y^t, with $t > 1$ (perfect power). Other related papers searched for Fibonacci numbers of the forms $px^2 + 1$, $px^3 + 1$ [12], $k^2 + k + 2$ [7], $p^a \pm p^b + 1$ [8], $p^a \pm p^b$ [9], $y^t \pm 1$ [2] and $q^k y^t$ [3]. Also, in 1993, Pethő and Tichy [11] proved that there are only finitely many Fibonacci numbers of the form $p^a + p^b + p^c$, with p prime. However, their proof uses the finiteness of solutions of S-unit equations, and as such is ineffective. Very recently, the authors [10] found all Fibonacci and Lucas numbers of the form $y^a + y^b + y^c$, with $2 \le y \le 9$.

In this paper, we are interested in Fibonacci and Lucas numbers which are sum of three perfect powers of some prescribed distinct bases. More precisely, our results are the following

Theorem 1.1. *The only solutions of the Diophantine equation*

$$F_n = 2^a + 3^b + 5^c \tag{1}$$

in integers n, a, b, c, with $0 \le \max\{a, b\} \le c$ are

$$(n, a, b, c) \in \{(4, 0, 0, 0),\ (6, 1, 0, 1)\}.$$

Theorem 1.2. *The only solutions of the Diophantine equation*

$$L_n = 2^a + 3^b + 5^c \tag{2}$$

in integers n, a, b, c, with $0 \le \max\{a, b\} \le c$ are

$$(n, a, b, c) \in \{(2, 0, 0, 0),\ (4, 0, 0, 1),\ (7, 0, 1, 2)\}.$$

2. Auxiliary results. First, we recall the well-known Binet's formulae for Fibonacci and Lucas sequences:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$. These formulas allow to deduce the bounds

$$\alpha^{n-2} \le F_n \le \alpha^{n-1} \text{ and } \alpha^{n-1} \le L_n \le 2\alpha^n,$$

which hold for all $n \ge 1$.

The next tools are related to the transcendental approach to solve Diophantine equations. First, we use a lower bound for a linear form in logarithms *à la Baker* and such a bound was given by the following result due to Laurent [6, Corollary 2] with $m = 24$ and $C_2 = 18.8$.

Lemma 1. *Let α_1, α_2 be real algebraic numbers, with $|\alpha_j| \ge 1$, b_1, b_2 be positive integer numbers and*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_j be real numbers such that

$$\log A_j \ge \max\{h(\alpha_j), |\log \alpha_j|/D, 1/D\}, j \in \{1, 2\},$$

2010 Mathematics Subject Classification. Primary 11B39, 11J86.

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doi: 10.3792/pjaa.89.47

where D is the degree of the number field $\mathbf{Q}(\alpha_1,\alpha_2)$ over $\mathbf{Q}$. Define

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

If α_1,α_2 are multiplicatively independent, then

$$\log|\Lambda| \geq -18.8\cdot D^4(\max\{\log b' + 0.38, m/D, 1\})^2 \cdot \log A_1 \log A_2.$$

As usual, in the above statement, the *logarithmic height* of an ℓ-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{\ell}\left(\log|a| + \sum_{j=1}^{\ell}\log\max\{1,|\gamma^{(j)}|\}\right),$$

where a is the leading coefficient of the minimal polynomial of γ (over $\mathbf{Z}$) and $(\gamma^{(j)})_{1\leq j\leq \ell}$ are the conjugates of γ (over $\mathbf{Q}$).

After finding an upper bound on n which is general too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethő [4]. For a real number x, we use $\|x\| = \min\{|x-n| : n\in\mathbf{N}\}$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M\|\gamma q\|$, where μ is a real number. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m, n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

See Lemma 5, a.) in [4]. Now, we are ready to deal with the proofs of our results.

3. Proof of the Theorem 1.1. Combining Binet formula together with (2), we get

$$\frac{\alpha^n}{\sqrt{5}} - 5^c = 2^a + 3^b + \frac{\beta^n}{\sqrt{5}} > 0, \tag{3}$$

because $|\beta| < 1$ while $2^a \geq 1$. Thus

$$\frac{\alpha^n 5^{-c}}{\sqrt{5}} - 1 = \frac{2^a}{5^c} + \frac{3^b}{5^c} + \frac{\beta^n}{5^c\sqrt{5}}$$

yields

$$\left|\frac{\alpha^n 5^{-c}}{\sqrt{5}} - 1\right| < \frac{3}{5^{0.3c}},$$

where we use that $2 < \sqrt{5}, 3 < 5^{0.7}$ and $c \geq \max\{a,b\}$. Therefore,

$$|e^{\Lambda_F} - 1| < \frac{3}{5^{0.3c}}, \tag{4}$$

where $\Lambda_F = n\log\alpha - (2c+1)\log\sqrt{5}$. By (3), $\Lambda_F > 0$ and in particular $e^{\Lambda_F} \neq 1$. Thus $\Lambda_F < e^{\Lambda_F} - 1$ and so

$$\log\Lambda_F < \log 3 - 0.48c. \tag{5}$$

In order to apply Lemma 1, we take

$$\alpha_1 := \sqrt{5},\ \alpha_2 := \alpha,\ b_1 := 2c+1,\ b_2 := n.$$

For this choice, we have $D = 2$, $h(\alpha_1) = \log\sqrt{5} < 0.81$ and $h(\alpha_2) = (\log\alpha)/2 < 0.25$. In conclusion, $\log A_1 := 0.81$ and $\log A_2 := 0.25$ are suitable choices. We also obtain the estimate

$$\alpha^{n-2} < F_n = 2^a + 3^b + 5^c < 2\cdot 5^c,$$

which implies that $n < 3.4c + 3.5$ (as we know that $2^a + 3^b \leq 2^c + 3^c < 5^c$). Thus we have

$$b' = \frac{2c+1}{0.5} + \frac{n}{1.62} < 6.1c + 4.2.$$

As α and 5 are multiplicatively independent, we have, by Lemma 1, that

$$\log|\Lambda_F| > -58.97 \cdot (\max\{\log(6.1c+4.2) + 0.38, 11\})^2. \tag{6}$$

We now combine (5) and (6) to get

$$c < 122.86 \cdot (\max\{\log(6.1c+4.2) + 0.38, 11\})^2 + \log 3$$

and so $c < 17585$ and $n < 59793$.

Since $0 < \Lambda_F < 3/5^{0.3c}$, we can rewrite this as

$$0 < n\log\alpha - c\log 5 + \log(1/\sqrt{5}) < 3\cdot(1.6)^{-c}.$$

Since $c > (n-3.5)/3.4 > 0.29n - 1.03$, we obtain (dividing by $\log 5$)

$$0 < n\gamma - c + \mu < 3.1\cdot(1.14)^{-n}, \tag{7}$$

with $\gamma := \log\alpha/\log 5$ and $\mu := \log(1/\sqrt{5})/\log 5 = -1/2$.

We claim that γ is irrational. In fact, if $\gamma = p/q$, then $\alpha^{2q}\in\mathbf{Q}$, which is an absurdity. Let q_n be the denominator of the n-th convergent of the continued fraction of γ. Taking $M := 59793$, we have

$$q_{12} = 369777 > 6M$$

and then $\epsilon := \|\mu q_{12}\| - M\|\gamma q_{12}\| = 0.44198\ldots$. Note that the conditions to apply Lemma 2 are fulfilled for $A = 3.1$ and $B = 1.14$, and hence there is no solution to inequality (7) (and then no solution to the Diophantine equation (1)) for n in the range

$$\left[\left\lfloor \frac{\log(Aq_{12}/\epsilon)}{\log B} \right\rfloor + 1, M\right) = [113, 59793).$$

Thus $n \leq 112$ and the estimate $5^c < F_n \leq F_{112}$ yields $c \leq 33$.

In order to still decrease the upper bound for c, we note that $\nu_5(F_n - 2^a - 3^b) = c$. To get an upper bound for this 5-adic valuation, we need to exclude the trivial cases when $F_n - 2^a - 3^b = 0$ (e.g. $(n, a, b) = (5, 1, 1)$ giving an infinite valuation), because clearly they don't give any solution. Thus, `Mathematica` returns $\nu_5(F_n - 2^a - 3^b) \leq 6$, for $n \leq 112$, $0 \leq \max\{a, b\} \leq 33$. Therefore $c \leq 6$ and then $n \leq 17$.

Finally, we use a program written in `Mathematica` to find the solutions of Eq. (1) in the range $0 \leq \max\{a, b\} \leq c \leq 6$ and $n \leq 17$. Quickly, the program returns the following solutions

$$(n, a, b, c) \in \{(4, 0, 0, 0), (6, 1, 0, 1)\}.$$

This completes the proof. □

4. Proof of the Theorem 1.2. By combining Binet formula together with (2), we get

$$\alpha^n - 5^c = 2^a + 3^b - \beta^n > 0 \tag{8}$$

and similarly as in the proof of previous theorem, we obtain

$$|e^{\Lambda_L} - 1| < \frac{3}{5^{0.3c}},$$

where $\Lambda_L := n \log\alpha - c \log 5$. The estimates $\Lambda_L > 0$ and $\Lambda_L < e^{\Lambda_L} - 1$ lead to

$$\log|\Lambda_L| < \log 3 - 0.48c. \tag{9}$$

To apply Lemma 1, we take

$$D = 2,\ b_1 = c,\ b_2 = n,\ \alpha_1 = 5,\ \alpha_2 = \alpha.$$

We choose $\log A_1 = 1.61$ and $\log A_2 = 0.25$. So we get

$$b' = \frac{c}{0.5} + \frac{n}{3.22} < 3.1c + 0.8,$$

where we use $n < 3.4c + 2.5$, which is obtained from $\alpha^{n-1} < L_n < 2 \cdot 5^c$.

As α and 5 are multiplicatively independent, by Lemma 1 we get

$$\log|\Lambda_L| \geq -116.57 \cdot (\max\{\log(3.1c + 0.8) + 0.38, 11\})^2. \tag{10}$$

Now, we combine the estimates (9) and (10) to obtain

$$c < 242.86 \cdot (\max\{\log(3.1c + 0.8) + 0.38, 11\})^2 + 2.3. \tag{11}$$

Therefore inequality (11) gives $c \leq 34790$ and so $n \leq 118289$.

In this case, the reduction method is not useful for reducing the bounds. However, we use the following approach. First, note that $c = \nu_5(L_n - 2^a - 3^b)$. To get an upper bound for this 5-adic valuation, we also need to exclude the trivial cases when $L_n - 2^a - 3^b = 0$ (e.g. $(n, a, b) = (3, 0, 1)$), because it doesn't give any solution. Notice that contrarily to the Fibonacci case, the bounds for n, a and b are very large, more precisely $n \leq 118289$ and $a, b \leq 34790$. Thus, it roughly took for `Mathematica` 102 hours on 2.5 GHz Intel Core i5 4 GB Mac OSX to return $\nu_5(L_n - 2^a - 3^b) \leq 26$. Therefore, $c \leq 26$ and then $n \leq 90$.

To finish, we use again `Mathematica` to find the solutions of Eq. (2) in the range $0 \leq \max\{a, b\} \leq c \leq 26$ and $n \leq 90$. We get the following solutions

$$(n, a, b, c) \in \{(2, 0, 0, 0),\ (4, 0, 0, 1),\ (7, 0, 1, 2)\}.$$

This completes the proof. □

5. Final comments. We remark that we can use our approach to prove that if $(G_n)_n$ is an linear recurrence sequence (under some weak technical assumptions), then there are only finitely many solutions (and all of them are effectively computable) for the Diophantine equation

$$G_n = p_1^{a_1} + p_2^{a_2} + \cdots + p_k^{a_k},$$

in integers $n, a_1, \ldots, a_k$, with $n > 0$ and $0 \leq \max\{a_1, \ldots, a_{k-1}\} \leq a_k$, where $p_1, \ldots, p_k$ are distinct primes previously fixed. However, it is important to notice that for each choice of primes, this study brings a lot of particular techniques.

Acknowledgements. The first author is grateful to CNPq, DPP-UnB and FEMAT-Brazil for the financial support. The second author is supported in part by Purdue University North Central.

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