Nilpotent class field theory for manifolds

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Abstract: We introduce a nilpotent Hurewicz homomorphism for a topological manifold, following the Lie algebra method in nilpotent class field theory for local and number fields by H. Koch *et al.* Using Labute-Anick's results on the determination of the Lie algebra attached to the lower central series of a group, we present explicitly nilpotent class field theory for some two and three dimensional manifolds.

Key words: Nilpotent Hurewicz homomorphism; nilpotent class field theory; lower central series; graded Lie algebras.

Introduction. Let X be a path-connected topological manifold*) with a base point x_0. We shall simply write $\pi_1(X)$ for the fundamental group $\pi_1(X, x_0)$ and $H_1(X)$ for the 1st integral homology group $H_1(X, \mathbf{Z})$. The Hurewicz homomorphism

$$\pi_1(X) \longrightarrow H_1(X)$$

induces the isomorphism

$$\pi_1^{\mathrm{ab}}(X) \xrightarrow{\sim} H_1(X)$$

where $\pi_1^{\mathrm{ab}}(X)$ stands for the abelianization $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$. Let X^{ab} be the maximal abelian covering over X corresponding to the commutator subgroup $[\pi_1(X), \pi_1(X)]$ by Galois theory. Viewing $\pi_1^{\mathrm{ab}}(X)$ as the Galois group $\mathrm{Gal}(X^{\mathrm{ab}}/X)$, the inverse of the Hurewicz isomorphism

$$h_X \ : \ H_1(X) \xrightarrow{\sim} \mathrm{Gal}(X^{\mathrm{ab}}/X)$$

is seen as a prototype in topology of unramified class field theory in arithmetic.

In this note, we apply the Lie algebra method in nilpotent class field theory by Koch *et al.* ([AKL], [KKL]) to this topological setting to introduce the nilpotent Hurewicz homomorphism and describe the nilpotent tower of coverings of X corresponding to the lower central series of $\pi_1(X)$.

1. The nilpotent Hurewicz homomorphism. For a group G and a positive integer q, let $G^{(q)}$ be the q-th term of the lower central series of G defined inductively by

$$G^{(1)} := G, \quad G^{(q+1)} := [G^{(q)}, G],$$

where $[A, B]$ stands for the subgroup of G generated by $[a, b] := aba^{-1}b^{-1}$ $(a \in A, b \in B)$ for subgroups A, B of G. Let $\mathrm{Gr}(G)$ denote the graded Lie algebra over $\mathbf{Z}$ defined by

$$\mathrm{Gr}(G) := \bigoplus_{q \geq 1} \mathrm{Gr}_q(G), \quad \mathrm{Gr}_q(G) := G^{(q)}/G^{(q+1)},$$

where the Lie bracket on $\mathrm{Gr}(G)$ is given by the group commutator.

Let X be a path-connected topological manifold. Let X^{nil} be the maximal nilpotent covering over X corresponding to $\cap_{q \geq 1} \pi_1(X)^{(q)}$, and let $X^{(q)}$ denote the Galois covering over X corresponding to $\pi_1(X)^{(q)}$ for each $q \geq 1$. Then we have

$$\begin{aligned} \mathrm{Gr}_q(\pi_1(X)) &= \mathrm{Gr}_q(\mathrm{Gal}(X^{\mathrm{nil}}/X)) \\ &= \mathrm{Gal}(X^{(q+1)}/X^{(q)}) \quad \text{for } q \geq 1 \end{aligned}$$

and so

$$\mathrm{Gr}(\pi_1(X)) = \mathrm{Gr}(\mathrm{Gal}(X^{\mathrm{nil}}/X)).$$

On the other hand, let $T(H_1(X))$ be the non-associative tensor algebra on $H_1(X)$ over $\mathbf{Z}$:

$$\begin{aligned} T(H_1(X)) &:= \bigoplus_{q \geq 1} T_q(H_1(X)), \\ T_1(H_1(X)) &:= H_1(X), \\ T_2(H_1(X)) &:= H_1(X) \otimes H_1(X), \\ T_q(H_1(X)) &:= \bigoplus_{i+j=q} T_i(H_1(X)) \otimes T_j(H_1(X)). \end{aligned}$$

Here $\otimes$ is taken over $\mathbf{Z}$. Let $\mathcal{L}(H_1(X))$ be the Lie

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*) It is enough to assume that X is a path-connected, locally path-connected and semilocally simply-connected, to use Galois theory. We assume that X is a topological manifold, for simplicity and applications.

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algebra over $\mathbf{Z}$ defined to be the quotient algebra of $T(H_1(X))$ by the ideal $\mathcal{I}$ generated by elements of the form

$$a\otimes a,\quad (a\otimes b)\otimes c+(b\otimes c)\otimes a+(c\otimes a)\otimes b,\quad (a,b,c\in T(H_1(X))).$$

When we write $\overline{a}:=a \bmod \mathcal{I}$, the Lie bracket in $\mathcal{L}(H_1(X))$ is defined by $[\overline{a},\overline{b}]:=a\otimes b \bmod \mathcal{I}$ $(a,b\in T(H_1(X)))$. Since the natural map $H_1(X)\to\mathcal{L}(H_1(X))$ is injective, we denote $\overline{\alpha}$ by α if $\alpha\in H_1(X)$. Let $\mathcal{L}_q(H_1(X))$ be the image of $T_q(H_1(X))$ under the natural projection $T(H_1(X))\to\mathcal{L}(H_1(X))$. Then $\mathcal{L}(H_1(X))$ forms a graded Lie algebra:

$$\mathcal{L}(H_1(X))=\bigoplus_{q\geq 1}\mathcal{L}_q(H_1(X)).$$

The Hurewicz isomorphism h_X is then seen as

$$h_X\ :\ H_1(X)\stackrel{\sim}{\longrightarrow}\mathrm{Gr}_1(\mathrm{Gal}(X^{\mathrm{nil}}/X))\subset\mathrm{Gr}(\mathrm{Gal}(X^{\mathrm{nil}}/X)).$$

Then by the correspondence

$$a_1\otimes(a_2\otimes(\cdots(a_{q-1}\otimes a_q)\cdots))\mapsto[h_X(a_1),[h_X(a_2),[\cdots,[h_X(a_{q-1}),h_X(a_q)]\cdots]]]$$

$(a_1,\ldots,a_q\in H_1(X))$, we obtain a graded Lie algebra homomorphism

$$H_X\ :\ \mathcal{L}(H_1(X))\longrightarrow\mathrm{Gr}(\mathrm{Gal}(X^{\mathrm{nil}}/X)) \tag{1.1}$$

which extends the Hurewicz homomorphism h_X. We call H_X the *nilpotent Hurewicz homomorphism* for X.

The problem of nilpotent class field theory for a topological manifold X is then understood as the determination of the image and kernel of the nilpotent Hurewicz homomorphism H_X ([O]).

2. The Labute-Anick conditions. In this section, we recall some results due to Labute and Anick ([A], [L1]) on the determination of the Lie algebra attached to the lower central series of a group.

Suppose G is a finitely presented group, namely, there is a free group F on the letters $x_1,\cdots,x_n$ and the subgroup R of F generated normally by the words $r_1,\cdots,r_m$ of $x_1,\cdots,x_n$ so that G is given by

$$G=F/R=\langle x_1,\cdots,x_n\mid r_1=\cdots r_m=1\rangle.$$

Let $\mathrm{Gr}(F)=\bigoplus_{q\geq 1}\mathrm{Gr}_q(F)$, $\mathrm{Gr}_q(F):=F^{(q)}/F^{(q+1)}$ be the graded Lie algebra associated to the lower central series $\{F^{(q)}\}_{q\geq 1}$ of F. If ξ_i denotes $x_i \bmod F^{(2)}\in\mathrm{Gr}_1(F)$, then $\mathrm{Gr}(F)$ is a free Lie algebra on $\xi_1,\cdots,\xi_n$ ([S, Ch.IV,3]). For $x\in F\neq 1$, let $w(x)$ be the largest integer q so that $x\in F^{(q)}$ and call $x \bmod F^{(w(x)+1)}\in\mathrm{Gr}_{w(x)}(F)$ the *initial form* of x. Let $\rho_1,\cdots,\rho_m$ be the initial forms of $r_1,\cdots,r_m$ and let $I=(\rho_1,\cdots,\rho_m)$ be the ideal of $\mathrm{Gr}(F)$ generated by $\rho_1,\cdots,\rho_m$. The problem of the determination of $\mathrm{Gr}(G)$ here is understood as the question: when does the natural homomorphism $\mathrm{Gr}(F)\to\mathrm{Gr}(G)$ of graded Lie algebras induce an isomorphism $\mathrm{Gr}(F)/I\simeq\mathrm{Gr}(G)$?

A sufficient condition for this, due to Labute, is formulated as follows. Let $U(\mathrm{Gr}(F)/I)$ be the universal enveloping algebra of $\mathrm{Gr}(F)/I$. Then, $I/[I,I]$ is a $U(\mathrm{Gr}(F)/I)$-module via the adjoint action. Then, we have the following

Theorem 2.1 ([L1]). *Assume the condition* (L): $\mathrm{Gr}(F)/I$ *is a free* $\mathbf{Z}$*-module and* $I/[I,I]$ *is a free* $U(\mathrm{Gr}(F)/I)$*-module on* $\rho_1,\ldots,\rho_m$.
Then, we have $\mathrm{Gr}(F)/I=\mathrm{Gr}(G)$.
Furthermore, under the condition (L), *we have* $U(\mathrm{Gr}(F)/I)=\mathrm{Gr}(\mathbf{Z}[G])$, *where* $\mathrm{Gr}(\mathbf{Z}[G]):=\bigoplus_{q\geq 0}J^q/J^{q+1}$ *is the graded algebra filtered by the powers of the augmentation ideal* J *of the group algebra* $\mathbf{Z}[G]$.

Next, we recall a useful sufficient condition for (L) due to D. Anick ([A]). First, we note that we can regard $\rho_1,\cdots,\rho_m$ as homogeneous elements in the universal enveloping algebra $U(\mathrm{Gr}(F))$ of $\mathrm{Gr}(F)$, which is identified with the polynomial algebra $\mathbf{Z}\langle\xi_1,\cdots\xi_n\rangle$ of non-commuting variables $\xi_1,\cdots,\xi_n$ over $\mathbf{Z}$. In general, we call a set of homogeneous elements $\{\rho_1,\cdots\rho_m\}$ in the polynomial algebra $A=k\langle\xi_1,\cdots,\xi_n\rangle$ over a field k *inert* if the quotient map $A\to B:=A/(\rho_1,\cdots,\rho_m)$ induces the injection $\mathrm{Tor}_2^A(k,k)\to\mathrm{Tor}_2^B(k,k)$ and isomorphism $\mathrm{Tor}_j^A(k,k)\to\mathrm{Tor}_j^B(k,k)$ for $j>2$. Then, we have the following

Theorem 2.2 ([A]). *Notations being as above, the followings are equivalent:*
(1) *the condition* (L) *holds,*
(2) *the image of* $\{\rho_1,\cdots,\rho_m\}$ *in* $\mathbf{F}_p\langle\xi_1,\cdots,\xi_n\rangle$ *is inert for any prime* p,
(3) *the homogeneous* q*-component of* $\mathrm{Gr}(F)/I$ *is a free* $\mathbf{Z}$*-module of rank* a_q *and the Poincaré series of* $U(\mathrm{Gr}(F)/I)$ *is given by*

$$\prod_{q\geq 1}(1-t^q)^{-a_q}=\left(1-nt+\sum_{j=1}^{m}t^{w(r_j)}\right)^{-1}.$$

A final sufficient condition for the inertness, due to also Anick, is combinatorial. Choose a total ordering on $\{\xi_1, \cdots, \xi_n\}$. For any non-zero homogeneous element ρ of degree d in $\mathbf{F}_p\langle \xi_1, \cdots, \xi_n\rangle$, write

$$\rho = \sum_{j=1}^{N} c_j w_j, \quad c_j \in \mathbf{Z}$$

where $\{w_1, \cdots, w_N\}$ is a complete set of the monomials in $\xi_1, \cdots, \xi_n$ which inherits the lexicographic order, and we call the *high term* of ρ is the highest w_j with $c_j \neq 0$. Then, we have the following

Theorem 2.3 ([A]). *Let $\{\rho_1, \cdots, \rho_m\}$ be any set of non-zero homogeneous elements in $\mathbf{F}_p\langle \xi_1, \cdots, \xi_n\rangle$ and w_j the high term of ρ_j. Suppose the condition* (C)*: no w_i is a submonomial of any w_j for $i \neq j$ and no w_i overlaps with any w_j. Namely, $w_i = uv, w_j = vw$ cannot occur unless $v = 1$ or $u = w = 1$.*
Then, $\{\rho_1, \cdots, \rho_m\}$ is inert in $\mathbf{F}_p\langle \xi_1, \cdots, \xi_n\rangle$.

3. Nilpotent class field theory. We now turn back to our problem on the nilpotent topological class field theory. We assume that the fundamental group $\pi_1(X)$ of a topological manifold X has a finite presentaion given by

$$\pi_1(X) = F/R = \langle x_1, \cdots, x_n \mid r_1 = \cdots r_m = 1\rangle.$$

Let $\xi_i := x_i \bmod F^{(2)}$ and ρ_i the initial form of r_i in $\mathrm{Gr}(F)$, and let α_i be the image of x_i in $H_1(X) = \mathrm{Gr}_1(\pi_1(X))$.

Theorem 3.1. *Notations being as above, suppose that the condition* (L) *in Theorem 2.1 is satisfied. Then the nilpotent Hurewicz homomorphism is surjective. If we assume further that $H_1(X)$ is a free $\mathbf{Z}$-module with basis $\alpha_1, \ldots, \alpha_n$, then the kernel of H_X is the ideal of $\mathcal{L}(H_1(X))$ generated by the elements corresponding to $\rho_1, \ldots, \rho_m$ under the identification $\mathcal{L}(H_1(X)) = \mathrm{Gr}(F)$.*

Proof. Under the condition (L), we have $\mathrm{Gr}(\pi_1(X)) = \mathrm{Gr}(F)/I$ when I is the ideal of $\mathrm{Gr}(F)$ generated by $\rho_1, \ldots, \rho_m$. By the definition of the nilpotent Hurewicz homomorphism H_X (1.1), we have $H_X(\alpha_i) = x_i \bmod I$ $(1 \leq i \leq n)$. Since $\mathrm{Gr}(F)$ is generated by x_i, H_X is surjective. The latter assertion is obvious. □

Example 3.2 (Closed surface). Let Σ_g be an oriented closed surface of genus $g \geq 0$. The fundamental group $\pi_1(\Sigma_g)$ has the presentation

$$\pi_1(\Sigma_g) = \langle x_1, \cdots, x_{2g} \mid r = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] = 1\rangle$$

and the homology group $H_1(\Sigma_g)$ is a free $\mathbf{Z}$-module with basis $\alpha_1, \cdots, \alpha_{2g}$ corresponding to the loops $x_1, \cdots, x_{2g}$.

Theorem 3.2.1. *Notations being as above, the nilpotent Hurewicz homomorphism H_{Σ_g} induces the isomorphism*

$$\mathcal{L}(H_1(\Sigma_g)) \Big/ \left(\sum_{i=1}^{2g-1} [\alpha_i, \alpha_{i+1}]\right) \xrightarrow{\sim} \mathrm{Gr}(\mathrm{Gal}(\Sigma_g^{\mathrm{nil}}/\Sigma_g)),$$

and $\mathrm{Gr}_q(\mathrm{Gal}(\Sigma_g^{\mathrm{nil}}/\Sigma_g)) = \mathrm{Gal}(\Sigma_g^{(q+1)}/\Sigma_g^{(q)})$ is a free $\mathbf{Z}$-module of rank

$$\frac{1}{q} \sum_{d|q} \mu\left(\frac{q}{d}\right)(\lambda^d + \eta^d)$$

where $1 - 2gt + t^2 = (1 - \lambda t)(1 - \eta t)$ and $\mu(\cdot)$ is the Möbius function.

Proof. Since the initial form of $r = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ is given by $\rho = [\xi_1, \xi_2] + \cdots + [\xi_{2g-1}, \xi_{2g}]$ where $[\xi_i, \xi_{i+1}] = \xi_i \xi_{i+1} - \xi_{i+1}\xi_i$, the condition (C) is satisfied for any p and so the condition (L) holds by Theorem 2.3. Further $H_1(X)$ is a free $\mathbf{Z}$-module with basis $\alpha_1, \ldots, \alpha_n$. Hence Theorem 3.1 yields the first assertion. By Theorem 2.2 and $w(r) = 2$, $\mathrm{Gr}_q(\mathrm{Gal}(\Sigma_g^{\mathrm{nil}}/\Sigma_g))$ is a free $\mathbf{Z}$-module whose rank a_q is given by

$$\prod_{q \geq 1} (1 - t^q)^{-a_q} = (1 - 2gt + t^2)^{-1} = \{(1 - \lambda t)(1 - \eta t)\}^{-1}.$$

Using $-\log(1 - s) = \sum_{n \geq 1} s^n/n$, we obtain

$$\sum_{d|q} d a_d = \lambda^q + \eta^q,$$

from which the second assertion follows ([S, Ch.IV,4]). □

Example 3.3 (Link complement). Let $L = K_1 \cup \cdots \cup K_n$ be a pure braid link with n-component knots in the 3-sphere S^3 $(n \geq 2)$ and $E_L := S^3 \setminus L$ the complement of L. The fundamental group $G_L := \pi_1(E_L)$, called the link group of L, has the presentation

$$G_L = \langle x_1, \cdots, x_n \mid [x_1, y_1] = \cdots = [x_n, y_n] = 1\rangle$$

where x_i and y_i represent the meridian and longitude around K_i, $1 \leq i \leq n$, and one of the relators is redundant ([B, Theorem 2.2]). The homology group $H_1(E_L)$ is a free $\mathbf{Z}$-module with basis $\alpha_1, \cdots, \alpha_n$, the meridian classes around K_i's. Let D be the diagram of L whose vertices are K_i's with K_i and K_j joined by an edge of weight l_{ij} = the linking number of K_i and K_j.

Theorem 3.3.1. *Notations being as above, assume further that the diagram D of L is connected mod p for any prime number p. Then the nilpotent Hurewicz homomorphism H_{E_L} induces the isomorphism*

$$\mathcal{L}(H_1(E_L))/(\gamma_1,\cdots,\gamma_{n-1}) \xrightarrow{\sim} \mathrm{Gr}(\mathrm{Gal}(E_L^{\mathrm{nil}}/E_L))$$

where $\gamma_i := \sum_{j\neq i} l_{ij}[\alpha_i,\alpha_j]$ $(1\le i\le n-1)$. *We also have*

$$\begin{aligned}\mathrm{Gr}_q(\mathrm{Gal}(E_L^{\mathrm{nil}}/E_L)) &= \mathrm{Gal}(E_L^{(q+1)}/E_L^{(q)})\\ &\simeq F(1)^{(q)}/F(1)^{(q+1)}\\ &\quad\times F(n-1)^{(q)}/F(n-1)^{(q+1)},\end{aligned}$$

where $F(m)$ is a free group of rank m. Hence $\mathrm{Gr}_q(\mathrm{Gal}(E_L^{\mathrm{nil}}/E_L))$ *is a free* $\mathbf{Z}$*-module whose rank* a_q *is given by*

$$a_q = \begin{cases} 1+b_q & \text{if } q=1,\\ b_q & \text{if } q>1,\end{cases} \quad \text{where}$$

$$b_q := \frac{1}{q}\sum_{d|q}\mu\left(\frac{q}{d}\right)(n-1)^d.$$

Proof. Anick ([A, Proposition 3.5]; See also [L2]) showed that, under the assumption on D, the initial form of $[x_i,y_i]$ is $\rho_i=\sum_{j\neq i} l_{ij}[\xi_i,\xi_j]$ of degree 2 and the condition (C) is satisfied for any p. Further $H_1(E_L)$ is a free $\mathbf{Z}$-module on $\alpha_1,\ldots,\alpha_n$. Hence Theorem 3.1 yields the first assertion. By Theorem 2.2 and $w([x_i,y_i])=2$, $\mathrm{Gr}_q(\mathrm{Gal}(E_L^{\mathrm{nil}}/E_L))$ is a free $\mathbf{Z}$-module whose a_q is given by

$$\begin{aligned}\prod_{q\ge1}(1-t^q)^{-a_q} &= (1-nt+(n-1)t^2)^{-1}\\ &= \{(1-t)(1-(n-1)t)\}^{-1},\end{aligned}$$

which is the Poincaré series of $U(\mathrm{Gr}(F(1))\oplus\mathrm{Gr}(F(n-1)))$. From this, the second assertion follows. □

Example 3.4 **(S^1-bundle over a torus).** For a positive integer e, we let

$$G := \left\{\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} \mid a,b,c\in\mathbf{R}\right\},$$

$$\Gamma_e := \left\{\begin{pmatrix}1 & l & n/e\\ 0 & 1 & m\\ 0 & 0 & 1\end{pmatrix} \mid l,m,n\in\mathbf{Z}\right\}$$

and

$$M_e := G/\Gamma_e.$$

The 3-dimensional manifold M_e is a principal S^1-bundle over the 2-dimensional torus T^2 with Euler class $e\in H^2(T^2,\mathbf{Z})=\mathbf{Z}$ whose fundamental group is given by

$$\begin{aligned}\pi_1(M_e)=\Gamma_e = \langle x_1,x_2,x_3 \mid [x_1,x_3]\\ = [x_2,x_3]=1, [x_1,x_2]=x_3^e\rangle\end{aligned}$$

where x_1, x_2 and x_3 are free words representing respectively

$$\gamma_1 := \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\ \gamma_2 := \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} \text{ and}$$

$$\gamma_3 := \begin{pmatrix}1&0&1/e\\0&1&0\\0&0&1\end{pmatrix}\in\Gamma_e.$$

(i) The case of $e=1$. For this case, we have

$$\begin{aligned}\pi_1(M_1)=\Gamma_1 &= \langle x_1,x_2,x_3 \mid [x_1,x_2]=x_3, [x_1,[x_1,x_2]]\\ &= [x_2,[x_1,x_2]]=1\rangle = \langle x_1,x_2 \mid [x_1,[x_1,x_2]]\\ &= [x_2,[x_1,x_2]]=1\rangle\end{aligned}$$

and the homology group $H_1(M_1)$ is a free $\mathbf{Z}$-module with basis α_1,α_2. Since the initial forms of $[x_1,[x_1,x_2]]$ and $[x_2,[x_1,x_2]]$ do not satisfy the condition (C), we cannot use Theorem 2.3. However we have the following

Theorem 3.4.1. *Notations being as above, the nilpotent Hurewicz homomorphism H_{M_1} induces the isomorphism*

$$\begin{aligned}&\mathcal{L}(H_1(M_1))/([\alpha_1,[\alpha_1,\alpha_2]],[\alpha_2,[\alpha_1,\alpha_2]])\\ &\quad\xrightarrow{\sim}\mathrm{Gr}(\mathrm{Gal}(M_1^{\mathrm{nil}}/M_1))\end{aligned}$$

and we have

$$\mathrm{Gr}_q(\mathrm{Gal}(M_1^{\mathrm{nil}}/M_1)) = \begin{cases}\mathbf{Z}^2 & \text{if } q=1,\\ \mathbf{Z} & \text{if } q=2,\\ 0 & \text{if } q>2.\end{cases}$$

Proof. Since Γ_1 is the nilpotent group with $\Gamma_1^{(2)}=\langle\gamma_3\rangle=\langle[\gamma_1,\gamma_2]\rangle=\mathbf{Z}$, $\Gamma_1^{(q)}=\{1\}$ $(q\ge3)$, the second assertion follows. Let $\mathcal{J}$ be the ideal of $\mathcal{L}(H_1(M_1))$ generated by $[\alpha_1,[\alpha_1,\alpha_2]],[\alpha_2,[\alpha_1,\alpha_2]]$. Then we have

$$\begin{aligned}&\mathcal{L}(H_1(M_1))/\mathcal{J}\\ &\quad= T(H_1(M_1))/([\alpha_1,[\alpha_1,\alpha_2]],[\alpha_2,[\alpha_1,\alpha_2]],\mathcal{I})\\ &\quad= H_1(M_1)\oplus\mathbf{Z}[\alpha_1,\alpha_2],\end{aligned}$$

from which the first assertion follows. □

(ii) The case $e\ge2$. For this case, the homology group $H_1(M_e)$ is given by $H_1(M_e)=\mathbf{Z}\alpha_1\oplus\mathbf{Z}\alpha_2\oplus\mathbf{Z}\alpha_3/e\mathbf{Z}\alpha_3=\mathbf{Z}^2\oplus\mathbf{Z}/e\mathbf{Z}$ and so it has torsion. So we cannot use Theorems 2.2, 2.3. However we have the following

Theorem 3.4.2. *Notations being as above, the nilpotent Hurewicz homomorphism H_{M_e} induces the isomorphism*

$$\mathcal{L}(H_1(M_e))/([\alpha_1,\alpha_3],[\alpha_2,\alpha_3],[\alpha_1,[\alpha_1,\alpha_2]],[\alpha_2,[\alpha_1,\alpha_2]]) \xrightarrow{\sim} \mathrm{Gr}(\mathrm{Gal}(M_e^{\mathrm{nil}}/M_e))$$

and we have

$$\mathrm{Gr}_q(\mathrm{Gal}(M_e^{\mathrm{nil}}/M_e)) = \begin{cases} \mathbf{Z}^2 \oplus \mathbf{Z}/e\mathbf{Z} & \text{if } q=1, \\ \mathbf{Z} & \text{if } q=2, \\ 0 & \text{if } q>2. \end{cases}$$

Proof. Since Γ_e is the nilpotent group with $\Gamma_e^{(2)} = \langle \gamma_3^e \rangle = \langle [\gamma_1,\gamma_2] \rangle = \mathbf{Z}$, $\Gamma_e^{(q)} = \{1\}$ $(q \geq 3)$, the second assertion follows. Let $\mathcal{J}$ be the ideal of $\mathcal{L}(H_1(M_e))$ generated by $[\alpha_1,\alpha_3],[\alpha_2,\alpha_3],[\alpha_1,[\alpha_1,\alpha_2]]$, $[\alpha_2,[\alpha_1,\alpha_2]]$. Then we have

$$\begin{aligned} &\mathcal{L}(H_1(M_e))/\mathcal{J} \\ &= T(H_1(M_e))/([\alpha_1,\alpha_3],[\alpha_2,\alpha_3],[\alpha_1,[\alpha_1,\alpha_2]],[\alpha_2,[\alpha_1,\alpha_2]],\mathcal{I}) \\ &= H_1(M_e) \oplus \mathbf{Z}[\alpha_1,\alpha_2], \end{aligned}$$

from which the first assertion follows. □

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