Multiple zeta values and zeta-functions of root systems

By Yasushi KOMORI,*) Kohji MATSUMOTO**) and Hirofumi TSUMURA***)

(Communicated by Masaki KASHIWARA, M.J.A., May 12, 2011)

Abstract: We propose the viewpoint that the r-ple zeta-function of Euler-Zagier type can be regarded as a specialization of the zeta-function associated with the root system of C_r type. From this viewpoint, we can see that Zagier's well-known formula for multiple zeta values (MZVs) coincides with Witten's volume formula associated with a sub-root system of C_r type. Based on this observation, we generalize Zagier's formula and also give analogous results which correspond to a sub-root system of B_r type. We announce those results as well as some relevant results for partial multiple zeta values.

Key words: Multiple zeta-values; root systems; witten zeta-functions.

1. Zeta-functions of root systems. The aim of this article is to announce our theory based on the observation that the Euler-Zagier r-ple sum

$$\zeta_r(s_1,\ldots,s_r)=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}}$$

(where $s_1,\ldots,s_r$ are complex variables; see Hoffman [3], Zagier [20]) can be regarded as a specialization of the zeta-function of the root system of C_r type. The details will appear elsewhere.

First we prepare notations. For the details of basic facts about root systems and Weyl groups, see [2,4,5].

Let V be an r-dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite irreducible reduced root system, and $\Psi=\{\alpha_1,\ldots,\alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta=\Delta_+\coprod\Delta_-$. Let $Q=Q(\Delta)$ be the root lattice, $Q^\vee$ the coroot lattice, $P=P(\Delta)$ the weight lattice, $P^\vee$ the coweight lattice, and P_{++} the set of integral strongly dominant weights respectively defined by

$$Q=\bigoplus_{i=1}^{r}\mathbf{Z}\,\alpha_i,\qquad Q^\vee=\bigoplus_{i=1}^{r}\mathbf{Z}\,\alpha_i^\vee,$$
$$P=\bigoplus_{i=1}^{r}\mathbf{Z}\,\lambda_i,\qquad P^\vee=\bigoplus_{i=1}^{r}\mathbf{Z}\,\lambda_i^\vee,$$
$$P_{++}=\bigoplus_{i=1}^{r}\mathbf{N}\,\lambda_i,$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of $\Psi^\vee$ and Ψ satisfying $\langle\alpha_i^\vee,\lambda_j\rangle=\delta_{ij}$ (Kronecker's delta) and $\langle\lambda_i^\vee,\alpha_j\rangle=\delta_{ij}$ respectively.

Let $\sigma_\alpha: V\to V$ be the reflection with respect to a root $\alpha\in\Delta$ defined by

$$\sigma_\alpha: v\mapsto v-\langle\alpha^\vee,v\rangle\alpha.$$

For a subset $A\subset\Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha\in A$. In particular, $W=W(\Delta)$ is the Weyl group, and $\{\sigma_j:=\sigma_{\alpha_j}\mid 1\le j\le r\}$ generates W. For $w\in W$, denote $\Delta_w=\Delta_+\cap w^{-1}\Delta_-$. The zeta-function associated with Δ is defined by

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta)=\sum_{\lambda\in P_{++}}e^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},$$

where $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}\in\mathbf{C}^{|\Delta_+|}$ and $\mathbf{y}\in V$ (for the details, see [7–15]). This can be regarded as a multi-variable version of Witten zeta-functions formulated by Zagier [20] based on the work of Witten [18].

2. Fundamental formulas. In this section, we state several fundamental formulas which are certain extensions of our previous results given in [9,10,15].

2000 Mathematics Subject Classification. Primary 11M41; Secondary 40B05.

*) Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan.
**) Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya, Aichi 464-8602, Japan.
***) Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan.

doi: 10.3792/pjaa.87.103

Let $\mathscr{V}$ be the set of all bases $\mathbf{V} \subset \Delta_+$. Let $\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbf{Z}\beta^\vee$. Then we have $|Q^\vee/L(\mathbf{V}^\vee)| < \infty$. Fix $\phi \in V$ such that $\langle \phi, \mu_\beta^{\mathbf{V}}\rangle \neq 0$ for all $\mathbf{V} \in \mathscr{V}$ and $\beta \in \mathbf{V}$. If the root system Δ is of A_1 type, then we choose $\phi = \alpha_1^\vee$. We define a multiple generalization of the fractional part as

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle < 0). \end{cases}$$

Let $\mathbf{T} = \{t \in \mathbf{C} \mid |t| < 2\pi\}^{|\Delta_+|}$.

Definition 2.1. For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+} \in \mathbf{T}$ and $\mathbf{y} \in V$, we define

$$F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}} \left(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \right) \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \times \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \left(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta \{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \right),$$

which is independent of choice of ϕ.

Remark 2.2. In [10], $F(\mathbf{t},\mathbf{y};\Delta)$ is defined in a different way. The above is [10, Theorem 4.1].

For $\mathbf{v} \in V$, and a differentiable function f on V, let

$$(\partial_{\mathbf{v}} f)(\mathbf{y}) = \lim_{h\to 0} \frac{f(\mathbf{y}+h\mathbf{v}) - f(\mathbf{y})}{h}$$

and for $\alpha \in \Delta_+$,

$$\mathfrak{D}_\alpha = \left.\frac{\partial}{\partial t_\alpha}\right|_{t_\alpha=0} \partial_{\alpha^\vee}.$$

Let $A = \{\nu_1,\ldots,\nu_N\} \subset \Delta_+$, and define

$$\mathfrak{D}_A = \mathfrak{D}_{\nu_N} \cdots \mathfrak{D}_{\nu_1}.$$

Further, let $A_j = \{\nu_1,\ldots,\nu_j\}$ $(1 \le j \le N-1)$, $A_0 = \emptyset$, and

$$\mathscr{V}_A = \{\mathbf{V} \in \mathscr{V} \mid \nu_{j+1} \notin \langle \mathbf{V} \cap A_j\rangle \mid (0 \le j \le N-1)\},$$

where $\langle\ \rangle$ denotes the linear span.

Theorem 2.3. *For $A = \{\nu_1,\ldots,\nu_N\} \subset \Delta_+$ and $\mathbf{t}_{\Delta_+\setminus A} = \{t_\alpha\}_{\alpha\in\Delta_+\setminus A}$, we have*

$$(\mathfrak{D}_A F)(\mathbf{t}_{\Delta_+\setminus A},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}_A} (-1)^{|A\setminus\mathbf{V}|} \times \left(\prod_{\gamma\in\Delta_+\setminus(\mathbf{V}\cup A)} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}\setminus A} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \right) \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \times \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \left(\prod_{\beta\in\mathbf{V}\setminus A} \frac{t_\beta \exp(t_\beta \{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \right),$$

which is independent of choice of the order of A. This function is holomorphic with respect to $\mathbf{t}_{\Delta_+\setminus A}$ around the origin.

Definition 2.4. For $A = \{\nu_1,\ldots,\nu_N\} \subset \Delta_+$ and $\mathbf{t}_{\Delta_+\setminus A} = \{t_\alpha\}_{\alpha\in\Delta_+\setminus A}$, we define $\mathcal{P}_{\Delta_+\setminus A}(\mathbf{k}_{\Delta_+\setminus A},\mathbf{y};\Delta)$ by

$$(\mathfrak{D}_A F)(\mathbf{t}_{\Delta_+\setminus A},\mathbf{y};\Delta) = \sum_{\mathbf{k}_{\Delta_+\setminus A}\in\mathbf{Z}_{\ge 0}^{|\Delta_+\setminus A|}} \mathcal{P}_{\Delta_+\setminus A}(\mathbf{k}_{\Delta_+\setminus A},\mathbf{y};\Delta) \prod_{\alpha\in\Delta_+\setminus A} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

Theorem 2.5. *For $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+}$ with $k_\alpha \in \mathbf{Z}_{\ge 1}$ $(\alpha \in \Delta_+ \setminus A)$, $k_\alpha = 0$ $(\alpha \in A)$, we have*

$$\sum_{w\in W} \left(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \right) \zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y};\Delta) = (-1)^{|\Delta_+|} \mathcal{P}_{\Delta_+\setminus A}(\mathbf{k}_{\Delta_+\setminus A},\mathbf{y};\Delta) \times \left(\prod_{\alpha\in\Delta_+\setminus A} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right) \tag{2.1}$$

provided all the series on the left-hand side absolutely converge.

Assume that Δ is not simply-laced. Then we have the disjoint union $\Delta = \Delta_l \cup \Delta_s$, where Δ_l is the set of all long roots and Δ_s is the set of all short roots. By applying Theorem 2.5 to $A = \Delta_l$ or Δ_s, we obtain the following theorem immediately, which is a generalization of the explicit volume formula proved in [15, Theorem 4.6].

Theorem 2.6. *Let $\Delta_1 = \Delta_l$ (resp. Δ_s), $\Delta_2 = \Delta_s$ (resp. Δ_l), and $\Delta_{j+} = \Delta_j \cap \Delta_+$ $(j = 1,2)$. For $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+}$ with $k_\alpha = k \in 2\mathbf{Z}_{\ge 1}$ $(\alpha \in \Delta_{1+})$, $k_\alpha = 0$ $(\alpha \in \Delta_{2+})$, and $\nu \in P^\vee/Q^\vee$, we have*

$$\zeta_r(\mathbf{k},\nu;\Delta) = \frac{(-1)^{|\Delta_+|}}{|W|} \mathcal{P}_{\Delta_{1+}}(\mathbf{k}_{\Delta_{1+}},\nu;\Delta) \left(\prod_{\alpha\in\Delta_{1+}} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \right).$$

3. Multiple zeta values.

Special values of Euler-Zagier sums when $s_1,\ldots,s_r$ are positive integers are usually called multiple zeta values (MZVs), and have been studied extensively. In the study of MZVs, the main target is to give non-trivial relations among them, in order to investigate the structure of the algebra generated by them (for the details, see Kaneko [6]). Here, we study MZVs from the viewpoint of zeta-functions of root systems. In our previous paper [14], we regarded MZVs as special values of zeta-functions of A_r type, and clarified the structure of the shuffle product procedure for MZVs. In this article, we regard MZVs as special values of zeta-functions of C_r type.

In the root system of C_r type, for $\Delta = \Delta(C_r)$, we have the disjoint union $\Delta_+^\vee = (\Delta_{l+})^\vee \cup (\Delta_{s+})^\vee$, where $\Delta_{l+} = \Delta_l \cap \Delta_+$, $\Delta_{s+} = \Delta_s \cap \Delta_+$, and

$$(\Delta_{l+})^\vee = \{\alpha_r^\vee,\ \alpha_{r-1}^\vee + \alpha_r^\vee, \ldots, \\ \alpha_2^\vee + \cdots + \alpha_r^\vee,\ \alpha_1^\vee + \cdots + \alpha_r^\vee\}.$$

Therefore by setting $s_\alpha = 0$ for $\alpha \in \Delta_{s+}$, we have

$$\zeta_r(\mathbf{s}, \mathbf{0}; \Delta) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{i=1}^{r} \frac{1}{(\sum_{j=r-i+1}^{r-1} m_j + m_r)^{s_i}},$$

which is exactly the Euler-Zagier sum $\zeta_r(s_1, \ldots, s_r)$. It is to be noted that some authors use the opposite order of summation in the definition of $\zeta_r(s_1, \ldots, s_r)$.

Corollary 3.1. *Let $\Delta = \Delta(C_r)$ and $\mathbf{2k}_{\Delta_{l+}} = (2k, \ldots, 2k) \in \mathbf{N}^r$ for any $k \in \mathbf{N}$,*

$$\zeta_r(2k, 2k, \ldots, 2k) \\ = \frac{(-1)^r}{2^r r!} \mathcal{P}_{\Delta_{l+}}(\mathbf{2k}_{\Delta_{l+}}, \mathbf{0}; \Delta) \frac{(2\pi i)^{2kr}}{\{(2k)!\}^r} \in \mathbf{Q} \cdot \pi^{2kr}.$$

Remark 3.2. The fact that $\zeta_r(2k, \ldots, 2k) \in \mathbf{Q} \cdot \pi^{2kr}$ was first proved by Zagier [20]. We emphasize that the above formula can be regarded as a kind of Witten's volume formula.

Let $\Delta = \Delta(C_2)$ be the root system of C_2 type. By Theorem 2.3, we have

$$(\mathfrak{D}_{\Delta_{s+}} F)(t_1, t_2, y_1, y_2; \Delta) \\ = 1 + \frac{t_1 t_2 e^{\{y_2\} t_1}}{(e^{t_1} - 1)(t_1 - t_2)} \\ + \frac{t_1 t_2 e^{\{y_2\} t_2}}{(e^{t_2} - 1)(-t_1 + t_2)} + \frac{t_1 t_2 e^{(1 - \{y_1 - y_2\}) t_1 + \{y_1\} t_2}}{(e^{t_1} - 1)(e^{t_2} - 1)} \\ - \frac{t_1 t_2 e^{(1 - \{2y_1 - y_2\}) t_1}}{(e^{t_1} - 1)(t_1 + t_2)} - \frac{t_1 t_2 e^{\{2y_1 - y_2\} t_2}}{(e^{t_2} - 1)(t_1 + t_2)} \\ = \sum_{k_1, k_2 = 1}^{\infty} \mathcal{P}_{\Delta_{l+}}(k_1, k_2, y_1, y_2; \Delta) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}.$$

Hence, by Corollary 3.1, we can compute $\zeta_2(2k, 2k)$ for $k \in \mathbf{N}$, though in this case we can also compute $\zeta_2(2k, 2k)$ by using the harmonic product formula for double zeta values

$$\zeta(s)\zeta(t) = \zeta_2(s, t) + \zeta_2(t, s) + \zeta(s + t).$$

In the general C_r case, considering the expansion of $(\mathfrak{D}_{\Delta_{s+}} F)(\mathbf{t}_{\Delta_{l+}}, \mathbf{0}; \Delta(C_r))$ similarly, we can systematically compute $\zeta_r(2k, \ldots, 2k)$. Moreover, considering the case $\nu \neq \mathbf{0}$ for $\zeta_r(\mathbf{s}, \nu; \Delta(C_r))$, we can give character analogues of Corollary 3.1 for multiple L-values, which were first proved by Yamasaki [19].

Next, we consider more general situation. In Theorem 2.5, we considered the sum over W on the left-hand side of (2.1). Here we consider the sum over a certain set of minimal coset representatives on the left-hand side of (2.1). Then we obtain the following result in the case of C_2.

Proposition 3.3. *For $p, q \in \mathbf{N}$, $p, q \geq 2$,*

$$(1 + (-1)^p)\zeta_2(p, q) + (1 + (-1)^q)\zeta_2(q, p) \\ = 2 \sum_{j=0}^{[p/2]} \binom{p + q - 2j - 1}{q - 1} \zeta(2j)\zeta(p + q - 2j) \\ + 2 \sum_{j=0}^{[q/2]} \binom{p + q - 2j - 1}{p - 1} \zeta(2j)\zeta(p + q - 2j) \\ - \zeta(p + q).$$

Actually this is a special case of the previous result for zeta-functions of A_2 type given by the third-named author [17, Theorem 4.5] (see also [12, Theorem 3.1]). In particular when p and q are of different parity, we see that $\zeta_2(p, q) \in \mathbf{Q}[\{\zeta(j + 1) \mid j \in \mathbf{N}\}]$ which was first proved by Euler. For example, we have

$$\zeta_2(2, 3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5).$$

On the other hand, considering the case of C_3 type, we have the following result which is not included in our previous result for zeta-functions of A_3 type (see [12, Theorem 7.1]).

Theorem 3.4. *For $p, q, u \in \mathbf{N}_{\geq 2}$,*

$$(1 + (-1)^p)(1 + (-1)^u) \\ \times \{\zeta_3(p, q, u) + \zeta_3(p, u, q) + \zeta_3(u, p, q)\} \\ + (1 + (-1)^q)(1 + (-1)^u) \\ \times \{\zeta_3(u, q, p) + \zeta_3(q, u, p) + \zeta_3(q, p, u)\} \\ \in \mathbf{Q}[\{\zeta(j + 1) \mid j \in \mathbf{N}\}].$$

In particular when p is odd and both q and u are even, then

(3.1) $$\zeta_3(u,q,p)+\zeta_3(q,u,p)+\zeta_3(q,p,u) \in \mathbf{Q}[\{\zeta(j+1)\mid j\in\mathbf{N}\}].$$

Remark 3.5. Combining (3.1) and the harmonic product formula for triple zeta values, we have

$$\zeta_3(p,q,u)-\zeta_3(u,q,p) \in \mathbf{Q}[\{\zeta(j+1),\ \zeta_2(k,l+1)\mid j,k,l\in\mathbf{N}\}],$$

when p is odd and both q and u are even. This is a known fact given by Borwein et al. in the triple case (see [1, Theorem 3.1]).

4. The case of B_r type. As for the root system of B_r type, namely for $\Delta=\Delta(B_r)$, we see that

$$(\Delta_{s+})^\vee=\{\alpha_r^\vee,\ 2\alpha_{r-1}^\vee+\alpha_r^\vee,\ldots,\ 2\alpha_1^\vee+\cdots+2\alpha_{r-1}^\vee+\alpha_r^\vee\}.$$

By setting $s_\alpha=0$ for all $\alpha\in\Delta_{l+}$, we have

$$\zeta_r(\mathbf{s},\mathbf{0};B_r)=\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{i=1}^{r}\frac{1}{(2\sum_{j=r-i+1}^{r-1}m_j+m_r)^{s_i}},$$

which is a partial sum of $\zeta_r(\mathbf{s})$. From the viewpoint of zeta-functions of root systems, values of this function at positive integers can be regarded as the objects dual to MZVs, in the sense that B_r and C_r are dual. For example,

$$\zeta_2((0,s_1,0,s_2),\mathbf{0};B_2)=\sum_{m,n=1}^{\infty}\frac{1}{n^{s_1}(2m+n)^{s_2}},$$

$$\zeta_3((0,0,s_1,0,0,s_2,0,0,s_3),\mathbf{0};B_3) =\sum_{l,m,n=1}^{\infty}\frac{1}{n^{s_1}(2m+n)^{s_2}(2l+2m+n)^{s_3}}.$$

For simplicity, we denote $\zeta_2((0,s_1,0,s_2),\mathbf{0};B_2)$ by $\zeta_2^\sharp(s_1,s_2)$. Then, similarly to Proposition 3.3, we can prove the following

Proposition 4.1. *For* $p,q\in\mathbf{N}_{\geq 2}$,

$$\begin{aligned}(1+(-1)^p)\zeta_2^\sharp(p,q)+(1+(-1)^q)\zeta_2^\sharp(q,p) &=2\sum_{j=0}^{[p/2]}\frac{1}{2^{p+q-2j}}\binom{p+q-1-2j}{q-1}\zeta(2j)\zeta(p+q-2j)\\ &\quad+2\sum_{j=0}^{[q/2]}\frac{1}{2^{p+q-2j}}\binom{p+q-1-2j}{p-1}\zeta(2j)\zeta(p+q-2j)\\ &\quad-\zeta(p+q).\end{aligned}$$

Example 4.2. In Proposition 4.1, if p and q are of different parity, then

$$\zeta_2^\sharp(p,q)\in\mathbf{Q}[\{\zeta(j+1)\mid j\in\mathbf{N}\}].$$

For example, setting $(p,q)=(3,2)$, we have

$$\zeta_2^\sharp(2,3)=\sum_{m,n=1}^{\infty}\frac{1}{n^2(2m+n)^3} =-\frac{21}{32}\zeta(5)+\frac{3}{8}\zeta(2)\zeta(3).$$

Furthermore, similarly to Corollary 3.1, we obtain the following

Proposition 4.3. *For* $k\in\mathbf{N}$,

$$\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{i=1}^{r}\frac{1}{(2\sum_{j=r-i+1}^{r-1}m_j+m_r)^{2k}}\in\mathbf{Q}\cdot\pi^{2kr}.$$

5. Partial zeta values. In [16], we studied zeta-functions of weight lattices of compact connected semisimple Lie groups. We can prove analogues of Theorem 2.5 for those zeta-functions by a method similar to the above. For example, considering the cases of B_2, C_2, B_3 and C_3 types, we obtain the following results on partial double and triple zeta values.

Theorem 5.1. *For* $p\in\mathbf{N}$,

$$\sum_{\substack{m,n=1\\ n\equiv 1\ (mod\,2)}}^{\infty}\frac{1}{n^{2p}(2m+n)^{2p}}\in\mathbf{Q}\cdot\pi^{4p},$$

$$\sum_{\substack{m,n=1\\ m\equiv 1\ (mod\,2)}}^{\infty}\frac{1}{n^{2p}(m+n)^{2p}}\in\mathbf{Q}\cdot\pi^{4p},$$

$$\sum_{\substack{l,m,n=1\\ n\equiv 1\ (mod\,2)}}^{\infty}\frac{1}{n^{2p}(2m+n)^{2p}(2l+2m+n)^{2p}}\in\mathbf{Q}\cdot\pi^{6p},$$

$$\sum_{\substack{l,m,n=1\\ l\equiv n\ (mod\,2)}}^{\infty}\frac{1}{n^{2p}(m+n)^{2p}(l+m+n)^{2p}}\in\mathbf{Q}\cdot\pi^{6p}.$$

Example 5.2. We can explicitly compute, for example,

$$\sum_{\substack{m,n=1\\ m\equiv 1\ (\mathrm{mod}\,2)}}^{\infty}\frac{1}{n^6(m+n)^6}=\frac{1}{58060800}\pi^{12},$$

$$\sum_{\substack{m,n=1\\ m\equiv 1\ (\mathrm{mod}\,2)}}^{\infty}\frac{1}{n^8(m+n)^8}=\frac{17}{390168576000}\pi^{16}.$$

References

[1] J. M. Borwein *et al.*, Special values of multiple polylogarithms, Trans. Amer. Math. Soc. **353** (2001), no. 3, 907–941.

[2] N. Bourbaki, *Groupes et Algèbres de Lie, Chapi-*

tres 4*,* 5 *et* 6, Hermann, Paris, 1968.

[3] M. E. Hoffman, Multiple harmonic series, Pacific J. Math. **152** (1992), no. 2, 275–290.

[4] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, New York, 1972.

[5] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29, Cambridge Univ. Press, Cambridge, 1990.

[6] M. Kaneko, Multiple zeta values, Sūgaku **54** (2002), no. 4, 404–415 (English translation; Sugaku Expositions **18** (2005), 221–232).

[7] Y. Komori, K. Matsumoto and H. Tsumura, Zeta functions of root systems, in *The Conference on L-Functions*, L. Weng and M. Kaneko (eds.), World Sci. Publ., Hackensack, NJ, 2007, 115–140.

[8] Y. Komori, K. Matsumoto and H. Tsumura, Zeta and L-functions and Bernoulli polynomials of root systems, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), no. 5, 57–62.

[9] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras II, J. Math. Soc. Japan **62** (2010), no. 2, 355–394.

[10] Y. Komori, K. Matsumoto and H. Tsumura, On multiple Bernoulli polynomials and multiple L-functions of root systems, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 303–347.

[11] Y. Komori, K. Matsumoto and H. Tsumura, An introduction to the theory of zeta-functions of root systems, in *Algebraic and analytic aspects of zeta functions and L-functions*, MSJ Mem., 21 Math. Soc. Japan, Tokyo, 2010, 115–140.

[12] Y. Komori, K. Matsumoto and H. Tsumura, Functional relations for zeta-functions of root systems, in *Number Theory*: *Dreaming in Dreams — Proceedings of the* 5th *China-Japan Seminar*, T. Aoki, S. Kanemitsu and J.-Y. Liu (eds.), World Sci. Publ., Hackensack, NJ, 2010, 135–183.

[13] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras III. (to appear in *The Proceedings of the Conference on Multiple Dirichlet Series and Applications to Automorphic Forms* (*Edinburgh,* 2008), D. Bump *et al.* (eds.), arXiv:0907.0955).

[14] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semi-simple Lie algebras IV, Glasg. Math. J. **53** (2011), no. 1, 185–206.

[15] Y. Komori, K. Matsumoto and H. Tsumura, Shuffle products for multiple zeta values and partial fraction decompositions of zeta-functions of root systems. (to appear in Math. Zeitschrift, arXiv:0908.0670).

[16] Y. Komori, K. Matsumoto and H. Tsumura, Zeta-functions of weight lattices of compact connected semisimple Lie groups, arXiv:1011.0323.

[17] H. Tsumura, On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, Math. Proc. Cambridge Philos. Soc. **142** (2007), no. 3, 395–405.

[18] E. Witten, On quantum gauge theories in two dimensions, Comm. Math. Phys. **141** (1991), no. 1, 153–209.

[19] Y. Yamasaki, Evaluations of multiple Dirichlet L-values via symmetric functions, J. Number Theory **129** (2009), no. 10, 2369–2386.

[20] D. Zagier, Values of zeta functions and their applications, in *First European Congress of Mathematics, Vol. II* (*Paris,* 1992), A. Joseph *et al.* (eds.), Progr. Math., 120 Birkhäuser, Basel, 1994, 497–512.