

Expansions on special solutions of the first q -Painlevé equation around the infinity

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Abstract: The first q -Painlevé equation has a unique formal solution around the infinity. This series converges only for $|q| = 1$. If q is a root of unity, this series expresses an algebraic function. In cases that all coefficients are integers, it can be represented by generalized hypergeometric series.

Key words: q -Painlevé equation; hypergeometric series.

1. Introduction. In this short note, we show a strange phenomenon on an asymptotic series which satisfy the first q -Painlevé equation.

It is known that the continuous first Painlevé equation $y'' = 6y^2 + t$ has an asymptotic solution of the form $y \sim \pm\sqrt{-t/6}$ ($t \rightarrow -\infty$), which is called Boutroux's tritronquée solution [1–3]. But we do not know asymptotic behavior on generic or special solutions for q -Painlevé equations.

Sakai gives a q -analogue of the first q -Painlevé equation (q - P_1):

$$(1) \quad \bar{f}f^2 \underline{f} = t(1 - f).$$

in [5]. Here $f = f(t)$, $\bar{f} = f(tq)$, $\underline{f} = f(t/q)$ for $q \in \mathbf{C}^*$. It is known that q - P_1 reduces to the continuous first Painlevé equation when $q \rightarrow 1$.

Recently Nishioka showed that q - P_1 has not any solution which is reduced to the first order q -difference equation [4]. The author does not know any examples of solutions of q - P_1 and we might show the first example of special solutions of q - P_1 .

2. Asymptotic expansions. Assume that q - P_1 has a formal solution of the form

$$f = \sum_{n=l}^{\infty} a_n(q)t^{-n}$$

for a suitable integer l . It is easy to show that $l = 0$ and $a_0(q) = 1$.

Theorem 2.1. *There exists a unique formal solution of the form*

$$f = \sum_{n=0}^{\infty} a_n(q)t^{-n} \quad (a_0(q) \neq 0).$$

$$a_0(q) = 1, \quad a_1(q) = -1,$$

$$a_2(q) = \frac{1}{q} + 2 + q,$$

$$a_3(q) = -\frac{1}{q^3} - \frac{2}{q^2} - \frac{5}{q} - 6 - 5q - 2q^2 - q^3,$$

$$a_4(q) = \frac{1}{q^6} + \frac{2}{q^5} + \frac{5}{q^4} + \frac{10}{q^3} + \frac{16}{q^2} + \frac{23}{q} + 26 + 23q + 16q^2 + 10q^3 + 5q^4 + 2q^5 + q^6.$$

Set $\beta(n) = n(n-1)/2$. Then we have

$$a_n(q) = (-1)^n \sum_{j=-\beta(n)}^{\beta(n)} c_{n,j}q^j.$$

Here $c_{n,j} \in \mathbf{Z}^+$, $c_{n,j} = c_{n,-j}$. Moreover f is a divergent series in case $0 < |q| < 1$ or $1 < |q|$.

Proof. It is evident that $a_n(q)$ satisfies a recurrence relation

$$a_n(q) = -F_n(a_1(q), a_2(q), \dots, a_{n-1}(q), q, 1/q).$$

The coefficients of the polynomial F_n with $(n+1)$ -variables are positive and

$$\begin{aligned} &F_n(a_1(q), \dots, a_{n-1}(q), q, 1/q) \\ &= F_n(a_1(q), \dots, a_{n-1}(q), 1/q, q). \end{aligned}$$

Since the leading term on q of F_n is

$$F_n = (2 + q^{n-1} + q^{1-n})a_{n-1}(q) + \dots,$$

the order of $a_n(q)$ on q is $\beta(n)$. Therefore f diverges for $0 < |q| < 1$ or $1 < |q|$. \square

We consider the case $|q| = 1$. Since $c_{n,j} = c_{n,-j}$, $a_n(q)$ is real for $|q| = 1$. If we set $q = e^{i\theta}$,

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$$\begin{aligned}
a_1(q) &= -1 \\
a_2(q) &= 2(1 + \cos \theta) \\
a_3(q) &= 2(3 + 5 \cos \theta + 2 \cos 2\theta + \cos 3\theta) \\
a_4(q) &= 4 \cos^2 \frac{\theta}{2} (5 + 16 \cos \theta + 4 \cos 2\theta \\
&\quad + 8 \cos 3\theta + 2 \cos 5\theta)
\end{aligned}$$

and

$$|a_n(q)| \leq (-1)^n a_n(1).$$

We show that $\{a_n(1)\}$ is nothing but a generalized hypergeometric series. We set

$$\begin{aligned}
G_1(t) &:= {}_4F_3\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}; \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; -\frac{256}{27t}\right), \\
G_2(t) &:= {}_4F_3\left(\frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}; \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; -\frac{256}{27t}\right), \\
G_3(t) &:= {}_4F_3\left(\frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}; \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; -\frac{256}{27t}\right).
\end{aligned}$$

Theorem 2.2. *When $|q| = 1$, f converges at least for $|t| > 256/27$. In case $q^N = 1$, f is algebraic. Iff $q = e^{k\pi i/3}$ ($k = 0, 1, 2, 3, 4, 5$) or $q = \pm i$, $a_n(q)$ are integers for any $n = 1, 2, 3, 4, \dots$*

In case $q = 1$, $f^4 = t(1 - f)$ and $f = G_1(t)$.

In case $q = -1$, $f^2(2 - f)^2 = t(1 - f)$ and $f = 1 - G_2(t^2)/t$.

In case $q = e^{2\pi i/3}$, $f^4 = t(1 - f)(1 - 3f + 3f^2)$ and

$$f = G_1(t^3) - \frac{1}{t} G_2(t^3) + \frac{1}{t^2} G_3(t^3).$$

In case $q = i$, f satisfies a complicated algebraic relation and

$$f = 1 - \frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right) G_1(t^4) - \frac{2}{t^3} G_2(t^4) + \frac{2}{t^4} G_3(t^4).$$

We can prove the theorem by direct calculation. It is open to represent $\{a_n(e^{\pi i/3})\}$ as a hyper-

geometric series, since it may satisfy a higher order equation.

3. Conclusion. We can consider the similar series for other q -Painlevé equations. For the continuous Painlevé equations except the sixth, generic asymptotic series around the infinity are always divergent. When the series are convergent, the solutions should be rational or confluent hypergeometric.

For q -Painlevé equations, asymptotic series might be also divergent for $|q| \neq 1$ except rational solutions and hypergeometric-type solutions. But the case $|q| = 1$ is interesting in the study of q -Painlevé equations.

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