Large deviations for the $\nabla\varphi$ interface model with self potentials

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(Communicated by Masaki KASHIWARA, M.J.A., March 12, 2009)

Abstract: The aim of this paper is to study, in the framework of the $\nabla\varphi$ interface model, the macroscopic behavior of microscopic interfaces under the finite volume Gibbs measures with self potentials, especially by establishing the large deviation principle. We consider the case where the self potential depends on the position and on both macroscopic and microscopic heights of the interfaces as well. The assumption on the upper bound for the self potential required by [10] is relaxed.

Key words: Large deviation principle; interface model; self potential.

1. Introduction and result. In this paper, we are interested in a macroscopic behavior of microscopic interfaces distributed under the finite volume Gibbs measures. In general, the interfaces are hypersurfaces which separate different phases like vapor and water. It is known that, at the macroscopic level, the most probable shape of a crystal surrounded by an interface having a definite total volume is characterized as a minimizer of the total surface tension and such shape is called Wulff shape. Mathematically, this can be shown as a consequence of large deviation principle. We prove the large deviations for the $\nabla\varphi$ interface model under the scaling limit for microscopic interfaces with self potentials. A survey on the $\nabla\varphi$ interface model is in [8], while a review of the results on the Ising model together with some explanations on the physical background can be found in [2].

The large deviation principle for the $\nabla\varphi$ interface model was first studied by Ben Arous and Deuschel [1]. They considered the Gibbs measure with quadratic potential having 0-boundary conditions. Deuschel, Giacomin and Ioffe [5] generalized the results to the non-Gaussian setting under the 0-boundary conditions. Then, taking an effect of self potentials into account, Funaki and Sakagawa [10] extended them for the Gibbs measure added a weak self potential under general Dirichlet boundary conditions, but they required that the self potentials take values between two limits as the height variables tends to $\pm\infty$, i.e. the condition (W2) below with $\gamma = \alpha \vee \beta$.

In our case, the self potentials may depend on microscopic and macroscopic height variables, and also on the macroscopic position of the interfaces. The values of our self potentials may be larger than two limits of them as the height variable tends to $\pm\infty$. In other words, our self potentials are rather free from the upper bound and therefore admit a wide class of functions.

We now formulate our problem more precisely. Let D be a bounded domain in $\mathbf{R}^d$ with a piecewise Lipschitz boundary and set $D_N = ND \cap \mathbf{Z}^d$. The location of the interface is described by a height variable $\phi = \{\phi(x) \in \mathbf{R}; x \in D_N\}$, which measures the vertical distance between the interface and the reference hyperplane D_N. We denote $\partial^+ D_N = \{x \notin D_N; |x-y| = 1 \text{ for some } y \in D_N\}$ and $\overline{D_N} = D_N \cup \partial^+ D_N$.

The Hamiltonian of ϕ on D_N with a boundary condition $\psi = \{\psi(x); x \in \partial^+ D_N\}$ and a self potential S is given by

$$H_N^{\psi,S}(\phi) = H_N^{\psi}(\phi) + \sum_{x \in D_N} S\left(\frac{x}{N}, \frac{1}{N}\phi(x), \phi(x)\right),$$

where

$$H_N^{\psi}(\phi) = \sum_{x,y \in \overline{D_N}, |x-y|=1} V((\phi \vee \psi)(x) - (\phi \vee \psi)(y))$$

and $\phi \vee \psi$ is the height variable on $\overline{D_N}$ determined by $(\phi \vee \psi)(x) = \phi(x)$ for $x \in D_N$ and $= \psi(x)$ for $x \in \partial^+ D_N$. The interaction potential V satisfies the following three conditions:

2000 Mathematics Subject Classification. Primary 60F10, 60K35.

doi: 10.3792/pjaa.85.31

(V1) (*smoothness*) $V \in C^2(\mathbf{R})$,
(V2) (*symmetry*) $V(\eta) = V(-\eta)$ for every $\eta \in \mathbf{R}$,
(V3) (*strict convexity*) there exist $c_-, c_+ > 0$ such that $c_- \le V''(\eta) \le c_+$ for every $\eta \in \mathbf{R}$.

The self potential $S : D \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is of the form $S(\theta, s, r) \equiv Q(\theta, s)W(r)$, where $Q : D \times \mathbf{R} \to [0, \infty)$ and $W : \mathbf{R} \to \mathbf{R}$. We assume the following conditions on Q and W, respectively:

(Q1) Q is non-negative, bounded and piecewise continuous,
(Q2) $|Q(\theta, s) - Q(\theta, s')| \le c(\theta)|s - s'|$, with $c : D \to [0, \infty)$ satisfying $\|c\|_{L^2(D)} < \infty$,
(W1) W is measurable,
(W2) the limits $\alpha = \lim_{r\to+\infty} W(r)$, $\beta = \lim_{r\to-\infty} W(r)$ exist, $\gamma = \sup_{r\in\mathbf{R}} W(r) < \infty$, and $W(r) \ge \alpha \wedge \beta$ for every $r \in \mathbf{R}$.

Remark 1.1. *Note that Funaki and Sakagawa* [10] *considered the case where S is decomposed into $S(\theta, s, r) = Q(\theta)W(r)$ and $\gamma = \alpha \vee \beta$ in the condition* (W2).

The macroscopic boundary condition will given by $g|_{\partial D}$ for $g \in C^\infty(\mathbf{R}^d)$. We assume the following conditions for the corresponding microscopic boundary condition $\psi \in \mathbf{R}^{\partial^+ D_N}$.

(PS1) $\max_{x\in\partial^+ D_N} |\psi(x)| \le CN$,
(PS2) $\sum_{x\in\partial^+ D_N} \left|\psi(x) - Ng(\frac{x}{N})\right|^{p_0} \le CN^d$ for some $C > 0$ and $p_0 > 2$.

The corresponding finite volume Gibbs measure on $\mathbf{R}^{D_N}$ is now defined by

$$\mu_N^{\psi,S}(d\phi) = \frac{1}{Z_N^{\psi,S}} \exp\{-H_N^{\psi,S}(\phi)\} \prod_{x\in D_N} d\phi(x),$$

where $Z_N^{\psi,S}$ is the normalization factor. The finite volume Gibbs measure without self potential is denoted by μ_N^{ψ}.

Our scaled random interface $\{h^N(\theta); \theta \in D\}$ is defined by a polilinear interpolation of the macroscopically scaled height variables, i.e., $h^N(\theta) = \frac{1}{N}\phi(x)$ for $\theta = \frac{x}{N}, x \in \overline{D_N}$ and

$$h^N(\theta) = \sum_{\lambda\in\{0,1\}^d} h^N\left(\frac{[N\theta] + \lambda}{N}\right) \times \left[\prod_{i=1}^d (\lambda_i\{N\theta_i\} + (1 - \lambda_i)(1 - \{N\theta_i\}))\right],$$

for general $\theta \in D$, where $[\cdot]$ and $\{\cdot\}$ denote the integral and the fractional parts, respectively (cf. [5]). We define another scaled profile $\{\overline{h}^N(\theta); \theta \in D\}$ by a step function, i.e., $\overline{h}^N(\theta) = \frac{1}{N}\phi([N\theta])$ for $\theta \in D$.

For $h \in H^1(D)$, define a surface free energy by

$$\Sigma(h) = \int_D \sigma(\nabla h(\theta))d\theta,$$

where $\sigma(u)$ is the surface tension with the tilt $u \in \mathbf{R}^d$ (cf. [5,8]).

Now we state our main theorem which establishes the large deviation principle for $\mu_N^{\psi,S}$ with weak self potentials in a wider class than those treated by Funaki and Sakagawa [10].

Theorem 1.1. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\mu_N^{\psi,S}$ satisfies the large deviation principle on $L^2(D)$ with speed N^d and the rate functional $I^S(h)$, that is, for every closed set $\mathcal{C}$ and open set $\mathcal{O}$ of $L^2(D)$ we have that*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mu_N^{\psi,S}(h^N \in \mathcal{C}) \le -\inf_{h\in\mathcal{C}} I^S(h),$$
$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mu_N^{\psi,S}(h^N \in \mathcal{O}) \ge -\inf_{h\in\mathcal{O}} I^S(h).$$

The functional $I^S(h)$ is given by

$$I^S(h) = \begin{cases} \Sigma^S(h) - \inf_{H_g^1(D)} \Sigma^S, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $H_g^1(D) = \{h \in H^1(D); h - g|_D \in H_0^1(D)\}$ and

$$(1) \quad \Sigma^S(h) = \Sigma(h) + \int_D Q(\theta, h(\theta))(\alpha 1_{\{h(\theta)>0\}} + \beta 1_{\{h(\theta)<0\}} + (\alpha \wedge \beta)1_{\{h(\theta)=0\}})d\theta.$$

It is well known that, once the large deviation principle is established, we immediately obtain the law of large numbers for h^N under $\mu_N^{\psi,S}$, that is, if the rate functional Σ^S has a unique minimizer h^* in $H_g^1(D)$, we have

$$\lim_{N\to\infty} \mu_N^{\psi,S}(\|h^N - h^*\|_{L^2(D)} > \delta) = 0$$

for every $\delta > 0$.

The law of large numbers under the situation that the rate functional has two distinct minimizers for the model motivated by the $\nabla\varphi$ interface model with self potentials in one dimension was discussed by [9] and [11]. The law of large number under the Gaussian Gibbs measures with δ-pinning, especially under the rate functional of the corresponding large deviation principle has two minimizers in one dimension was studied by [3]. In particular, they proved that two minimizers coexist in Free boundary case. The δ-pinning potential is defined from a certain limit of the square well pinning potential.

The large deviation principle for the square well pinning potential has not been proven yet. Dunlop *et al.* [7] first proved the localization under the Gaussian Gibbs measures with the square well pinning potential and 0-boundary conditions in two dimension. The result of [7] was extended for general convex potential by Deuschel and Velenik [6].

In the next section, we will give the Proof of Theorem 1.1.

2. Proof of Theorem 1.1. In this section, we only consider the case where $\beta \leq \alpha$. The case $\alpha < \beta$ can be treated in a similar way. We decompose the self potential S into $S = \gamma Q + Q(W - \gamma)$. Then, $\tilde{W} = W - \gamma$ satisfies the condition (W2′), which is the condition (W2) with $\gamma = 0$.

Remark 2.1. *If Q does not depend on $\phi(x)$, since the contribution of the first term γQ in $\exp\{-H_N^{\psi,S}(\phi)\}$ of $\mu_N^{\psi,S}$ cancels with the normalization factor, we can prove Theorem* 1.1 *in a similar way to the proof of* [10, Theorem 2.1]. *However, since Q of our self potential depends on $\phi(x)$, we cannot prove Theorem* 1.1 *by tracing the method used for the proof of* [10, Theorem 2.1]. *The following proposition recovers the thread.*

The following proposition is for the finite Gibbs measure $\mu_N^{\psi,\gamma Q}$ with $S = \gamma Q$.

Proposition 2.1. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\mu_N^{\psi,\gamma Q}$ satisfies the large deviation principle on $L^2(D)$ with speed N^d and the rate functional given by*

$$I^{\gamma Q}(h) = \begin{cases} \Sigma^{\gamma Q}(h) - \inf_{H_g^1(D)} \Sigma^{\gamma Q}, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$\Sigma^{\gamma Q}(h) = \Sigma(h) + \gamma \int_D Q(\theta, h(\theta)) d\theta.$$

To prove Proposition 2.1, we prepare the following Lemma.

Lemma 2.2. *Assume the conditions* (Q1) *and* (Q2) *on $Q(\theta, s)$. Let $g \in L^2(D)$ and $0 < \delta < 1$ be fixed. If $h^N \in B_2(g, \delta) = \{h \in L^2(D);\ \|h - g\|_{L^2(D)} < \delta\}$ for N large enough, then there exists some constant $C > 0$ such that*

$$\left| \gamma \frac{1}{N^d} \sum_{x \in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right) - \gamma \int_D Q(\theta, g(\theta)) d\theta \right| < C\delta,$$

for every N sufficiently large.

Proof. If $h^N \in B_2(g, \delta)$, then $\|\overline{h}^N - g\|_{L^2(D)} < C_1\delta + a_{N,k}$, where C_1 is a positive constant and $a_{N,k}$ tends to 0 as $N \to \infty$ and $k \to \infty$, see (3.2) in [10]. Therefore, by, (Q1) and (Q2), if $h^N \in B_2(g, \delta)$, then the left hand side of the desired inequality can be bounded by

$$\leq |\gamma|(C_1\delta + a_{N,k})\|c\|_{L^2(D)}|D| + |\gamma| C_2 \delta \leq C\delta,$$

for every N and k large enough, where C_2 and C are positive constants. □

Proof of Proposition 2.1.

Step1 (lower bound). Let $g \in L^2(D)$ and $\delta > 0$. Then, by Lemma 2.2 and the large deviation principle lower bound for μ_N^{ψ} (cf. [10, Proposition 3.1]), we have

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}} \mu_N^{\psi,\gamma Q}(h^N \in B_2(g,\delta)) &\geq - \inf_{h \in B_2(g,\delta)} I(h) - \gamma \int_D Q(\theta, g(\theta)) d\theta - C\delta \\ &\geq -\left\{ I(g) + \gamma \int_D Q(\theta, g(\theta)) d\theta \right\} - C\delta, \end{aligned}$$

where

$$I(h) = \begin{cases} \Sigma(h) - \inf_{H_g^1(D)} \Sigma, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise.} \end{cases}$$

is the rate functional of the large deviation principle for μ_N^{ψ}.

Now let us take an arbitrary open set $\mathcal{O}$ of $L^2(D)$. Then, for every $h \in \mathcal{O}$ and $\delta > 0$ such that $B_2(h, \delta) \subset \mathcal{O}$,

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}} \mu_N^{\psi,\gamma Q}(h^N \in \mathcal{O}) \geq -\left\{ I(h) + \gamma \int_D Q(\theta, h(\theta)) d\theta \right\} - C\delta. \end{aligned}$$

Letting $\delta \downarrow 0$, since $h \in \mathcal{O}$ is arbitrary, we have the lower bound

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}} \mu_N^{\psi,\gamma Q}(h^N \in \mathcal{O}) \geq - \inf_{h \in \mathcal{O}} \left\{ I(h) + \gamma \int_D Q(\theta, h(\theta)) d\theta \right\}. \tag{2}$$

Step2 (upper bound). Let $g \in L^2(D)$ and $\delta > 0$ be fixed. Then, by Lemma 2.2 and the large deviation principle upper bound for μ_N^{ψ} (cf. [10, Proposition 3.1]), we have

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}}\,\mu_N^{\psi,\gamma Q}(h^N\in B_2(g,\delta))$$
$$\le -\inf_{h\in\bar{B}_2(g,\delta)} I(h)-\gamma\int_D Q(\theta,g(\theta))d\theta + C\delta.$$

Then, by using the lower semi-continuity of $I(h)$, we see that for every $g\in L^2(D)$, there exists $\delta>0$ small enough such that

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}}\,\mu_N^{\psi,\gamma Q}(h^N\in B_2(g,\delta))$$
$$\le -\left\{I(g)+\gamma\int_D Q(\theta,g(\theta))d\theta\right\}.$$

The standard argument in the theory of large deviation principle (cf. [4]) yields the upper bound

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,\gamma Q}}{Z_N^{\psi}}\,\mu_N^{\psi,\gamma Q}(h^N\in \mathcal{C})$$
$$\le -\inf_{h\in\mathcal{C}}\left\{I(h)+\gamma\int_D Q(\theta,h(\theta))d\theta\right\} \tag{3}$$

for every compact set $\mathcal{C}$ of $L^2(D)$. However, the exponential tightness for $\mu_N^{\psi,\gamma Q}$ was proved in [10]. Thus, (3) holds for every closed set $\mathcal{C}$ of $L^2(D)$.

Taking $\mathcal{O}=\mathcal{C}=L^2(D)$ in (2) and (3), we have the conclusion. □

To prove Theorem 1.1, we also prepare the following lemmas. For $a,b,c,d,e\in\mathbf{R}$ and $g\in L^2(D)$, we set

$$\mathfrak{Q}(a,b,c;d,e):=$$
$$\int_D Q(\theta,g(\theta))\Big(a+b1_{\{g(\theta)\ge\sqrt{d}\}}+c1_{\{g(\theta)\le-\sqrt{e}\}}\Big)d\theta.$$

In particular, we write $\mathfrak{Q}(a,b,c;d):=\mathfrak{Q}(a,b,c;d,d)$.

Lemma 2.3. *Assume the conditions* (Q1), (Q2), (W1) *and* (W2′) *on* $S(\theta,s,r)=Q(\theta,s)W(r)$. *Let* $g\in L^2(D)$ *and* $0<\delta<1$ *be fixed. If* $h^N\in B_2(g,\delta)=\{h\in L^2(D);\|h-g\|_{L^2(D)}<\delta\}$ *with* N *large enough, then there exists some constant* $C>0$ *such that*

$$\sum_{x\in D_N} S\Big(\frac{x}{N},\frac{1}{N}\phi(x),\phi(x)\Big)-N^d\mathfrak{Q}(0,\alpha,\beta;\delta)$$
$$<CN^d\delta,$$

for every N *sufficiently large.*

Lemma 2.4. *Assume the conditions* (Q1), (Q2), (W1) *and* (W2′) *on* $S(\theta,s,r)=Q(\theta,s)W(r)$.
(1) *The functional* $\Sigma^S(h)$ *is lower semi-continuous on* $L^2(D)$.
(2) *Let* $\Sigma_-^S(h)$ *be the functional defined by* (1) *with* $1_{\{h(\theta)\le0\}}$ *replaced by* $1_{\{h(\theta)<0\}}$. *Then, for every open set* $\mathcal{O}$ *of* $L^2(D)$, *we have that*

$$\inf_{h\in\mathcal{O}}\Sigma^S(h)=\inf_{h\in\mathcal{O}}\Sigma_-^S(h).$$

Lemmas 2.3 and 2.4 are very similar to Lemmas 3.1 and 3.2 of [10], respectively. Therefore, we only give some remarks instead of completely proving the lemmas.

Remark 2.2. *In the proof of Lemma* 2.4 (2), *by replacing* $h^n(\theta)$ *which was defined in the proof of* [10, Lemma 3.2] *with*

$$h^n(\theta)=\begin{cases} h(\theta)-f^n(\theta), & \text{if } h(\theta)\le 0,\\ h(\theta), & \text{if } h(\theta)>0,\end{cases} \tag{4}$$

where $f^n\in C_0^\infty(D)$ *are functions such that* $f^n(\theta)\equiv\frac{1}{n}$ *on* $D_n=\{\theta\in D;\,\mathrm{dist}(\theta,\partial D)\ge\frac1n\}$ *and* $|\nabla f^n(\theta)|\le C$ *with* $C>0$, *we can get the conclusion in a similar way to the proof of* [10, Lemma 3.2 (2)]. *Moreover, the case* $\alpha<\beta$ *can be proved in a similar way replacing* (4) *by*

$$h^n(\theta)=\begin{cases} h(\theta), & \text{if } h(\theta)<0,\\ h(\theta)+f^n(\theta), & \text{if } h(\theta)\ge 0.\end{cases}$$

Proof of Theorem 1.1.

Step1 (lower bound). Let $g\in L^2(D)$ and $\delta>0$. Then, by Lemma 2.3 and the large deviation principle lower bound for $\mu_N^{\psi,\gamma Q}$ (Proposition 2.1), we have

$$\liminf_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}}\,\mu_N^{\psi,S}(h^N\in B_2(g,\delta))$$
$$\ge -\inf_{h\in B_2(g,\delta)} I^{\gamma Q}(h)-\mathfrak{Q}(0,\alpha-\gamma,\beta-\gamma;\delta)-C\delta$$
$$\ge -\{I^{\gamma Q}(g)+\mathfrak{Q}(0,\alpha-\gamma,\beta-\gamma;\delta)\}-C\delta.$$

Now let us take an arbitrary open set $\mathcal{O}$ of $L^2(D)$. Then, we have

$$\liminf_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}}\,\mu_N^{\psi,S}(h^N\in\mathcal{O})$$
$$\ge -\inf_{h\in\mathcal{O}}\left\{I^{\gamma Q}(h)+\int_D Q(\theta,h(\theta))\right.$$
$$\left.\times\big((\alpha-\gamma)1_{\{h(\theta)>0\}}+(\beta-\gamma)1_{\{h(\theta)<0\}}\big)d\theta\right\} \tag{5}$$

in a similar way to the proof of the lower bound of Proposition 2.1.

However, by Lemma 2.4 (2), one can replace $1_{\{h(\theta)<0\}}$ with $1_{\{h(\theta)\le0\}}$ on the right hand side of (5). Therefore, we get

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{O}) \geq -\inf_{h\in\mathcal{O}} \Sigma^S(h) + \inf_{h\in H_g^1(D)} \Sigma^{\gamma Q}(h). \tag{6}$$

Step2 (upper bound). Let $g \in L^2(D)$ and $\delta > 0$ be fixed. We define $L_N = N\{\theta \in D; g(\theta) > \sqrt{\delta}\} \cap \mathbf{Z}^d$. By the assumption (W2) on W, for every $\varepsilon > 0$ there exists $K = K_\varepsilon > 0$ such that $W(r) - \gamma \geq -(\alpha-\beta-\varepsilon)1_{\{r\leq K\}} + \alpha - \gamma - \varepsilon$ for any $r \in \mathbf{R}$. Therefore, we have

$$\begin{aligned}
&\exp\left\{-\sum_{x\in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right)(W(\phi(x)) - \gamma)\right\}\\
&\leq \exp\left\{(-\alpha+\gamma+\varepsilon)\sum_{x\in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right)\right\}\\
&\times \sum_{\Lambda\subset D_N}\prod_{x\in\Lambda}\left(e^{(\alpha-\beta-\varepsilon)Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)} - 1\right)1_{\{\phi(x)\leq K\}}.
\end{aligned}$$

Now, if $\phi(x) \leq K$ for $x \in L_N$, then $\frac{1}{N}\phi(x) - g(\frac{x}{N}) < -\frac{1}{2}\sqrt{\delta}$ for N large enough. Thus, since $\|\overline{h}^N - \overline{g}^N\|_{L^2(D)} < \frac{1}{C_0}(\delta + \|g - g^N\|_{L^2(D)})$, if $\phi(x) \leq K$ for every $x \in \Lambda \subset L_N$ on $\{h^N \in B_2(g,\delta)\}$, then we have for N large enough

$$\frac{2\delta^2}{C_0^2} > \frac{1}{N^d}\sum_{x\in D_N}\left(\frac{1}{N}\phi(x) - g\left(\frac{x}{N}\right)\right)^2 > \frac{|\Lambda|\delta}{4N^d},$$

namely, $|\Lambda| < \frac{8N^d\delta}{C_0^2}$, where $C_0 = C_0(d,p) > 0$ is the constant, see in [10, p.188]. Combining these all facts and Lemma 2.2

$$\begin{aligned}
&\frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in B_2(g,\delta))\\
&\leq e^{2CN^d\delta + N^d\mathfrak{Q}(-\alpha+\gamma+\varepsilon,0,\alpha-\beta-\varepsilon;0,\delta)}\\
&\times \left(e^{(\alpha-\beta-\varepsilon)\|Q\|_\infty} - 1\right)^{\frac{8N^d\delta}{C_0^2}} \mu_N^{\psi,\gamma Q}(h^N \in B_2(g,\delta))\\
&\times |\{\Lambda \subset L_N : |\Lambda| < 8N^d\delta C_0^{-2}\}|.
\end{aligned}$$

On the other hand, by using Stirling's formula, we see that

$$\begin{aligned}
&|\{\Lambda \subset L_N : |\Lambda| < 8N^d\delta C_0^{-2}\}|\\
&\leq \left(\frac{C}{\delta}\right)^{CN^d\delta} N^d(1+o(1))
\end{aligned}$$

as $N \to \infty$, for some constant $C > 0$ independent of N and δ (cf. [10, p.189]). Hence, by the large deviation principle upper bound for $\mu_N^{\psi,\gamma Q}$ (Proposition 2.1), we obtain

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in B_2(g,\delta))\\
&\leq -\inf_{h\in\bar{B}_2(g,\delta)} I^{\gamma Q}(h) + C(\delta)\\
&+ \mathfrak{Q}(-\alpha+\gamma+\varepsilon, 0, \alpha-\beta-\varepsilon; 0, \delta),
\end{aligned}$$

where $C(\delta)$ is a constant independent of N and converges to 0 as $\delta \to 0$. Then, by using the lower semi-continuity of $I^{\gamma Q}(h)$, we see that for every $g \in L^2(D)$ and $\varepsilon > 0$, there exists $\delta > 0$ small enough such that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in B_2(g,\delta))\\
&\leq -\left\{I^{\gamma Q}(g) + \mathfrak{Q}(\alpha-\gamma, 0, \beta-\alpha; 0)\right\} + \varepsilon|D|\|Q\|_\infty.
\end{aligned}$$

Therefore, the standard argument in the theory of large deviation principle yields

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{C})\\
&\leq -\inf_{h\in\mathcal{C}}\left\{I^{\gamma Q}(h) + \int_D Q(\theta, h(\theta))\right.\\
&\left.\times\left((\alpha-\gamma)1_{\{h(\theta)>0\}} + (\beta-\gamma)1_{\{h(\theta)\leq 0\}}\right)d\theta\right\},
\end{aligned}$$

for every compact set $\mathcal{C}$ of $L^2(D)$. The exponential tightness for $\mu_N^{\psi,S}$ can be proved in a similar way to those for μ_N^{ψ} (cf. Remark 4.1 of [10]). Thus, for every closed set $\mathcal{C}$ of $L^2(D)$, we get

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{C}) \leq -\inf_{h\in\mathcal{C}} \Sigma^S(h) + \inf_{H_g^1(D)} \Sigma^{\gamma Q}. \tag{7}$$

Taking $\mathcal{O} = \mathcal{C} = L^2(D)$ in (6) and (7), we have the conclusion. □

Acknowledgements. The author thanks Prof. Y. Tamura for many helpful discussions. The author also thanks Prof. T. Funaki for many helpful comments.

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