Meromorphic functions sharing four values

By Hong-Xun Yi*) and Xiao-Min Li**)

(Communicated by Heisuke Hironaka, M.J.A., Sept. 12, 2007)

Abstract: In this paper, we deal with the problem of uniqueness of meromorphic functions that share three values IM and a fourth value CM, and prove some results which answer a open question on uniqueness of meromorphic functions. Examples show that the conditions of theorems in this paper are necessary.

Key words: Nevanlinna theory; uniqueness of meromorphic functions; shared value; normal growth.

1. Introduction and main results. In this paper, a meromorphic function means meromorphic in the complex plane. We use the usual notations of Nevanlinna theory of meromorphic functions as explained in [1].

Let f be a nonconstant meromorphic function. The order of f, denoted $\sigma(f)$, is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

The lower order of f, denoted $\mu(f)$, is defined by

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If the order and the lower order of f are equal, that is $\sigma(f) = \mu(f)$, then f is called a function with normal growth.

Let f be a nonconstant meromorphic function. If the order of f is finite, we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(\log r) \quad (r \to \infty).$$

If the order of f is infinite, we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(\log(rT(r,f))) \quad (r \to \infty, r \notin E),$$

where E is a set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

Let f and g be two nonconstant meromorphic functions. If for some $a \in \mathbf{C} \cup \{\infty\}$, f and g have same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities) (see [2]).

In 1926, R. Nevanlinna [3] proved the following theorem.

Theorem A. *Let f and g be two distinct nonconstant meromorphic functions and a_j $(j = 1, 2, 3, 4)$ be four distinct values. If f and g share a_j $(j = 1, 2, 3, 4)$ CM, then f is a Möbius transformation of g. Furthermore, two of a_j $(j = 1, 2, 3, 4)$, say a_3 and a_4, are Picard exceptional values of f and g, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1976, L. Rubel asked the following question: whether CM can be replaced by IM in the hypothesis of Theorem A with the same conclusion or not? In 1979, G. G. Gundersen [4] gave a negative answer for this question by the following counterexample:

$$f = \frac{e^h + 1}{(e^h - 1)^2} \quad \text{and} \quad g = \frac{(e^h + 1)^2}{8(e^h - 1)},$$

where h is a nonconstant entire function. It is easy to verify that f and g share the four values $0, 1, \infty, -1/8$, where none of the four values are shared CM, and f is not a Möbius transformation of g. Other examples of meromorphic functions that share four values where none of the four values are shared CM, were given by N. Steinmetz [5] and M. Reinders [6].

On the other hand, G. G. Gundersen [7] proved the following result which is an improvement of Theorem A.

Theorem B. *If two distinct nonconstant meromorphic functions share two values CM and share two other values IM, then the functions share*

2000 Mathematics Subject Classification. Primary 30D35; Secondary 30D30.

*) Department of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China.
**) Department of Mathematics, Ocean University of China, Qingdao, Shandong 266071, People's Republic of China.

all four values CM (*hence the conclusions of Theorem A hold*).

Between these examples and Theorem B, G. G. Gundersen [10] posed the following open question, which is the long-standing one:

Gundersen's question: If two nonconstant meromorphic functions share three values IM and share a fourth value CM, then do the functions necessarily share all four values CM?

In resent years, improvements of Theorem B were made by G. G. Gundersen [10], E. Mues [11–13], M. Reinders [6,14], S. P. Wang [15], H. Ueda [16,17], H. X. Yi and C. T. Zhou [18], T. P. Czubiak and G. G. Gundersen [19], G. D. Qiu [20], J. P. Wang [21,22], B. Huang [23,24], B. Huang and J. Y. Du [25], K. Ishizaki [26], P. Li [27], W. H. Yao [28], and other authors (see [2]), but Gundersen's question is still open. Gundersen's question is the main open question in the theory of meromorphic functions that share four values. This question appears to be difficult (see [10]). In this paper we will do a step in this direction.

It is easy to prove that meromorphic function with two Picard exceptional values is a function with normal growth (see [2, Theorem 1.42 and Theorem 1.44]). From Theorem A, we know that two distinct nonconstant meromorphic functions sharing four values CM are functions with normal growth. It is natural to ask the following open question:

Open question: Must f and g be functions with normal growth, if f and g are two distinct nonconstant meromorphic functions sharing three values IM and sharing a fourth value CM?

In this paper we give a positive answer to this question. In fact, we shall prove the following theorem.

Theorem 1. *Let f and g be two distinct nonconstant meromorphic functions and a_j ($j = 1, 2, 3, 4$) be four distinct values. If f and g share a_1, a_2, a_3 IM and a_4 CM, then f and g are functions with normal growth, f and g have the same order, and the order of f and g is a positive integer or infinite.*

From Theorem 1, we immediately obtain the following theorems on uniqueness of meromorphic functions.

Theorem 2. *Let f and g be two nonconstant meromorphic functions sharing three values IM and sharing a fourth value CM. If f and g are not functions with normal growth, then $f \equiv g$.*

Theorem 3. *Let f and g be two nonconstant meromorphic functions sharing three values IM and sharing a fourth value CM. If the order of f and g is neither a positive integer nor infinite, then $f \equiv g$.*

The following examples show that the conclusion of Theorem 1 can occur, and that the condition "f and g are not functions with normal growth" in Theorem 2 is necessary, the condition "order of f and g is neither a positive integer nor infinite" in Theorem 3 is necessary.

Example 1. Let $f = e^z$, $g = e^{-z}$. It is easy to verify that f and g share $0, 1, \infty, -1$ CM and that $\mu(f) = \mu(g) = \sigma(f) = \sigma(g) = 1$, however $f \not\equiv g$.

Example 2. Let $f = \frac{2e^h}{e^h - 1}$, $g = \frac{2}{1-e^h}$, where h is a transcendental entire function. It is easy to verify that f and g share $0, 1, \infty, 2$ CM and that $\mu(f) = \mu(g) = \sigma(f) = \sigma(g) = \infty$, however $f \not\equiv g$.

2. Some Lemmas.

Lemma 1 (see [8] or [2, Theorem 2.16]). *Let f and g be nonconstant rational functions. If f and g share four distinct values IM, then $f \equiv g$.*

Lemma 2 (see [2, Corollary of Theorem 1.5]). *Let f be a nonconstant meromorphic function. Then f is a rational function if and only if*

$$\liminf_{r\to\infty} \frac{T(r, f)}{\log r} < \infty.$$

Lemma 3 (see [9] or [2, Theorem 1.19]). *Let $T_1(r)$ and $T_2(r)$ be two nonnegative and nondecreasing real functions defined in $r > r_0 > 0$. If*

$$T_1(r) = O(T_2(r)) \quad (r \to \infty, r \notin E),$$

where E is a set with finite linear measure, then

$$\limsup_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log^+ T_2(r)}{\log r}$$

and

$$\liminf_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \le \liminf_{r\to\infty} \frac{\log^+ T_2(r)}{\log r},$$

which mean that the order and the lower of $T_1(r)$ are not greater than the order and the lower of $T_2(r)$, respectively.

Lemma 4 (see [2, Theorem 1.42 and Theorem 1.44]). *Let f be a nonconstant meromorphic function and a, b be two distinct finite values. If a and b are Picard exceptional values of f, then*

$$f = \frac{ae^h - b}{e^h - 1},$$

where h is a nonconstant entire function. If h is a polynomial of degree γ, then $\mu(f)=\sigma(f)=\gamma$. If h is a transcendental entire function, then $\mu(f)=\sigma(f)=\infty$. In both cases, f is a function with normal growth.

Lemma 5 (see [1, Theorem 2.2] or [2, Theorem 1.7])**.** *Let f be a nonconstant meromorphic function and $k\geq 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f).$$

Lemma 6 (see [10, Lemma 1] or [2, Theorem 4.4])**.** *Let f and g be two distinct nonconstant meromorphic functions, and let a_1, a_2, a_3 and a_4 be four distinct complex numbers. If f and g share a_1, a_2, a_3 and a_4 IM, then*

(i) $T(r,f)=T(r,g)+S(r,f)$;

(ii) $\sum_{j=1}^{4}\overline{N}(r,\frac{1}{f-a_j})=2T(r,f)+S(r,f)$.

Lemma 7 (see [10, Lemma 3] or [2, Theorem 4.4])**.** *Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3 and a_4 IM, where $a_4=\infty$. Then the following statements hold*:

(i) *$N_1(r,0,f')=S(r,f)$ and $N_1(r,0,g')=S(r,f)$, where $N_1(r,0,f')$ and $N_1(r,0,g')$ "count" respectively only those points in $N(r,0,f')$ and $N(r,0,g')$ which do not occur when $f(z)=g(z)=a_i$ for some $i=1,2,3$.*

(ii) *For $i=1,2,3,4$, let $N_2(r,a_i)$ refer only to those a_i-points that are multiple for both f and g and "count" each such point the number of times of the smaller of the two multiplicities. Then*

$$\sum_{i=1}^{4}N_2(r,a_i)=S(r,f).$$

Lemma 8 (see [2, Lemma 4.3])**.** *Suppose that $f(z)$ is a nonconstant meromorphic function, and $P(f)=a_0f^p+a_1f^{p-1}+\cdots+a_p\,(a_0\neq 0)$ is a polynomial in f with degree p and coefficients a_j $(j=0,1,2,\cdots,p)$ are constants, suppose furthermore that b_j $(j=1,2,\cdots,q)\,(q>p)$ are distinct finite values. Then*

$$\frac{P(f)}{(f-b_1)(f-b_2)\cdots(f-b_q)}=\sum_{j=1}^{q}\frac{A_j}{f-b_j}$$

where $A_j\,(j=1,2,\cdots,q)$ are nonzero constants, and

$$m\left(r,\frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=S(r,f).$$

Let f and g be two nonconstant meromorphic functions sharing the value a IM. We denote by $\overline{N}_{(p,q)}(r,a)$ the reduced counting function of those common zeros of $f-a$ and $g-a$ such that a is taken by f with multiplicity p, and by g with multiplicity q.

Lemma 9. *Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3 and a_4 IM, where $a_4=\infty$. Then*

$$\frac{1}{7}T(r,f)\leq\sum_{j=1}^{3}\sum_{k=1}^{6}\overline{N}_{(1,k)}(r,a_j)+\sum_{j=1}^{3}\sum_{k=2}^{6}\overline{N}_{(k,1)}(r,a_j)+S(r,f).\tag{2.1}$$

Proof. From Lemma 6 (i) we have

$$T(r,f)=T(r,g)+S(r,f).\tag{2.2}$$

By Lemma 7 (ii) we obtain

$$\sum_{j=1}^{3}N_2(r,a_j)=S(r,f).\tag{2.3}$$

From (2.2) and (2.3) we get for $j=1,2,3$

(2.4)

$$\begin{aligned}\overline{N}(r,a_j)&=\sum_{k=1}^{\infty}\overline{N}_{(1,k)}(r,a_j)+\sum_{k=2}^{\infty}\overline{N}_{(k,1)}(r,a_j)+S(r,f)\\&=\sum_{k=1}^{6}\overline{N}_{(1,k)}(r,a_j)+\sum_{k=7}^{\infty}\overline{N}_{(1,k)}(r,a_j)\\&\quad+\sum_{k=2}^{6}\overline{N}_{(k,1)}(r,a_j)+\sum_{k=7}^{\infty}\overline{N}_{(k,1)}(r,a_j)+S(r,f)\\&\leq\sum_{k=1}^{6}\overline{N}_{(1,k)}(r,a_j)+\frac{1}{7}N\left(r,\frac{1}{g-a_j}\right)\\&\quad+\sum_{k=2}^{6}\overline{N}_{(k,1)}(r,a_j)+\frac{1}{7}N\left(r,\frac{1}{f-a_j}\right)+S(r,f)\\&\leq\sum_{k=1}^{6}\overline{N}_{(1,k)}(r,a_j)+\frac{1}{7}T(r,g)\\&\quad+\sum_{k=2}^{6}\overline{N}_{(k,1)}(r,a_j)+\frac{1}{7}T(r,f)+S(r,f)\\&\leq\sum_{k=1}^{6}\overline{N}_{(1,k)}(r,a_j)+\sum_{k=2}^{6}\overline{N}_{(k,1)}(r,a_j)\\&\quad+\frac{2}{7}T(r,f)+S(r,f).\end{aligned}$$

By the second fundamental theorem, we have

$$T(r,f) \le \sum_{j=1}^{3} \overline{N}(r,a_j) + S(r,f). \tag{2.5}$$

From (2.4) and (2.5) we get (2.1). □

3. Proof of Theorem 1. Since $f \not\equiv g$, by Lemma 1 we know that f and g are transcendental meromorphic functions. From (2.2) and Lemma 3, we obtain

$$\mu(f) = \mu(g), \quad \sigma(f) = \sigma(g). \tag{3.1}$$

Suppose that $\mu(f) = \infty$. Noting that $\mu(f) \le \sigma(f)$, we have $\mu(f) = \sigma(f) = \infty$. Thus, the conclusions of Theorem 1 hold. In the following we suppose $\mu(f) < \infty$.

Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, $a_3 = c$ and $a_4 = \infty$. Put

$$\begin{aligned}\Phi := \frac{f''}{f'} - \frac{f'}{f} - \frac{f'}{f-1} - \frac{f'}{f-c} \\ - \frac{g''}{g'} + \frac{g'}{g} + \frac{g'}{g-1} + \frac{g'}{g-c}.\end{aligned} \tag{3.2}$$

By Lemma 5 and (2.2), we have

$$m(r,\Phi) = S(r,f). \tag{3.3}$$

Since f and g share a_1, a_2, a_3 IM and a_4 CM, by simply calculating we can see that Φ is analytic at any point z such that $f(z) = g(z) = a_i$ for some $i = 1,2,3,4$. Thus, from (2.2) and Lemma 7 (i) we get

$$\begin{aligned}N(r,\Phi) &\le N_1(r,0,f') + N_1(r,0,g') \\ &= S(r,f).\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we can easily deduce

$$T(r,\Phi) = S(r,f). \tag{3.5}$$

Noting that $\mu(f) < \infty$, from (3.5) we can get

$$\liminf_{r\to\infty} \frac{T(r,\Phi)}{\log r} < \infty. \tag{3.6}$$

From (3.6) and Lemma 2 we can see that Φ is a rational function. From (3.2) we can deduce

$$\Phi = P_1 + \sum_{j=1}^{q} \frac{m_j}{z - z_j}, \tag{3.7}$$

where $q\,(\ge 0)$, $m_j\,(1 \le j \le q)$ are integers, and $z_j\,(1 \le j \le q)$ are those points such that $f'(z_j) = 0$ and $g'(z_j)(f(z_j) - a_i)(g(z_j) - a_i) \ne 0\,(i = 1,2,3)$, or $g'(z_j) = 0$ and $f'(z_j)(f(z_j) - a_i)(g(z_j) - a_i) \ne 0$ $(i = 1,2,3)$, or $f'(z_j) = 0$ and $g'(z_j) = 0$ with the different multiplicities, but $(f(z_j) - a_i)(g(z_j) - a_i) \ne 0$ for $i = 1,2,3$. By (3.7) and integrating two sides of (3.2), we can easily deduce

$$\frac{f'g(g-1)(g-c)}{g'f(f-1)(f-c)} = Q(z)e^P, \tag{3.8}$$

where $Q(z) = \prod_{j=1}^{q} (z - z_j)^{m_j}$ is a rational function, and $P = \int_0^z P_1(\eta)\,d\eta + A$ is a polynomial, A is a constant. Set

$$H := Q(z)e^P, \tag{3.9}$$

Then we have

$$H = \frac{f'g(g-1)(g-c)}{g'f(f-1)(f-c)}. \tag{3.10}$$

Assume that z_0 is a point such that $f(z_0) = a$ with multiplicity p and $g(z_0) = a$ with multiplicity q, where $a \in \{a_1, a_2, a_3\}$. From (3.10) we obtain

$$H(z_0) = \frac{p}{q}. \tag{3.11}$$

We discuss the following two cases.

Case 1. Suppose that $H \equiv C$, where C is a nonzero constant. By Lemma 9 we know that at least one of

$$\begin{gathered}\sum_{k=1}^{6} \overline{N}_{(1,k)}(r,a_j) + \sum_{k=2}^{6} \overline{N}_{(k,1)}(r,a_j) \\ \ne S(r,f) \quad (j = 1,2,3)\end{gathered}$$

must occur. Without loss of generality, we assume that

$$\sum_{k=1}^{6} \overline{N}_{(1,k)}(r,0) + \sum_{k=2}^{6} \overline{N}_{(k,1)}(r,0) \ne S(r,f). \tag{3.12}$$

From (3.12) we know that at least one of

$$\begin{aligned}&\overline{N}_{(1,1)}(r,0) \ne S(r,f), \\ &\overline{N}_{(1,k)}(r,0) \ne S(r,f) \quad (k = 2,3,\cdots,6)\end{aligned}$$

and

$$\overline{N}_{(k,1)}(r,0) \ne S(r,f) \quad (k = 2,3,\cdots,6)$$

must occur. We distinguish the following three subcases.

Subcase 1.1. Suppose that $\overline{N}_{(1,1)}(r,0) \ne S(r,f)$. Then there exists a point z_0 such that z_0 is a simple zero of both f and g. From (3.11) we have $C = 1$. By (3.10) we obtain

$$\frac{f'}{f(f-1)(f-c)} \equiv \frac{g'}{g(g-1)(g-c)}. \tag{3.13}$$

Noting that f and g share 0, 1, c IM and ∞ CM, from (3.13) we can easily see that f and g share 0, 1, c and ∞ CM. By Theorem A and Lemma 4, we can obtain the conclusions of Theorem 1.

Subcase 1.2. Suppose that $\overline{N}_{(1,k)}(r,0) \neq S(r,f)$ $(k = 2, 3, \cdots, 6)$. Then there exists a point z_0 such that z_0 is a simple zero of f, a zero of g of order k, where $2 \leq k \leq 6$. From (3.11) we have $C = \frac{1}{k}$. By (3.10) we obtain

$$\frac{kf'}{f(f-1)(f-c)} \equiv \frac{g'}{g(g-1)(g-c)}. \tag{3.14}$$

Noting that f and g share 0, 1, c IM, from (3.14) we can easily see that for the common zeros of $f-a$ and $g-a$, where $a \in \{0, 1, c\}$, if a is taken by f with multiplicity p, and by g with multiplicity q, then $\frac{p}{q} = \frac{1}{k}$.

Let

$$\phi_1 := \frac{kf'}{f(f-1)} - \frac{g'}{g(g-1)}. \tag{3.15}$$

It is easy to see that ϕ_1 is a entire function. By Lemma 5 and Lemma 8, from (2.2) and (3.15) we obtain $m(r, \phi_1) = S(r,f)$. Thus,

$$T(r, \phi_1) = S(r,f). \tag{3.16}$$

Noting that $\mu(f) < \infty$, from (3.16) we can get

$$\liminf_{r\to\infty} \frac{T(r,\phi_1)}{\log r} < \infty. \tag{3.17}$$

From (3.17) and Lemma 2 we can see that ϕ_1 is a polynomial. From (3.15) we can deduce

$$\frac{(f-1)^k g}{f^k (g-1)} = e^{\phi}, \tag{3.18}$$

where $\phi = \int_0^z \phi_1(\eta)\, d\eta + A$ is a polynomial, A is a constant. By (2.2) and (3.18) we have

$$T(r, e^{\phi}) \leq T\left(r, \frac{(f-1)^k}{f^k}\right) + T\left(r, \frac{g}{g-1}\right) + S(r,f)$$

$$= (k+1)T(r,f) + S(r,f). \tag{3.19}$$

Again by Lemma 3, we obtain

$$\mu(e^{\phi}) \leq \mu(f). \tag{3.20}$$

From (3.18) we obtain

$$\frac{(f-1)^k}{f^k} = e^{\phi} \cdot \frac{g-1}{g}. \tag{3.21}$$

From (3.21) we get

$$kT(r,f) \leq T(r, e^{\phi}) + T(r,g) + S(r,f). \tag{3.22}$$

By (2.2) and (3.22) we obtain

$$T(r, e^{\phi}) \geq (k-1)T(r,f) + S(r,f). \tag{3.23}$$

Noting f is a transcendental, from (3.23) we know that ϕ is a nonconstant polynomial. Thus,

$$\mu(e^{\phi}) = \sigma(e^{\phi}) = \gamma, \tag{3.24}$$

where γ is the degree of ϕ. By Lemma 3 and (3.23), we obtain

$$\sigma(f) \leq \sigma(e^{\phi}). \tag{3.25}$$

Noting that $\mu(f) \leq \sigma(f)$, from (3.20), (3.24) and (3.25) we obtain the conclusions of Theorem 1.

Subcase 1.3. Suppose that $\overline{N}_{(k,1)}(r,0) \neq S(r,f)$ $(k = 2, 3, \cdots, 6)$. Then there exists a point z_0 such that z_0 is a simple zero of g, a zero of f of order k, where $2 \leq k \leq 6$. Similar to Subcase 1.2, we can obtain the conclusions of Theorem 1.

Case 2. Suppose that H is not a constant. By (2.2) and (3.10) and Lemma 6 (ii) we have

$$\begin{aligned} T(r,H) &\leq T\left(r, \frac{f'}{f(f-1)(f-c)}\right) \\ &\quad + T\left(r, \frac{g'}{g(g-1)(g-c)}\right) + S(r,f) \\ &\leq 2\sum_{j=1}^{3} \overline{N}(r, a_j) + S(r,f) \\ &\leq 4T(r,f) + S(r,f). \end{aligned} \tag{3.26}$$

Again by Lemma 3, we obtain

$$\mu(H) \leq \mu(f). \tag{3.27}$$

From (3.11) we obtain

$$\begin{aligned} &\sum_{j=1}^{3}\sum_{k=1}^{6} \overline{N}_{(1,k)}(r, a_j) + \sum_{j=1}^{3}\sum_{k=2}^{6} \overline{N}_{(k,1)}(r, a_j) \\ &\quad \leq \sum_{k=1}^{6} \overline{N}\left(r, \frac{1}{H - \frac{1}{k}}\right) + \sum_{k=2}^{6} \overline{N}\left(r, \frac{1}{H-k}\right) \\ &\quad \leq 11T(r,H) + O(1). \end{aligned} \tag{3.28}$$

From (2.1) and (3.28) we get

$$T(r,f) \leq 77T(r,H) + S(r,f). \tag{3.29}$$

Noting f is transcendental, from (3.29) we know that H is transcendental. Again from (3.9) we know that P is a nonconstant polynomial. Thus,

$$\mu(H) = \sigma(H) = \gamma, \tag{3.30}$$

where γ is the degree of P. By Lemma 3 and (3.29), we obtain

$$\sigma(f) \leq \sigma(H). \tag{3.31}$$

Noting that $\mu(f) \leq \sigma(f)$, from (3.27), (3.30) and (3.31) we obtain the conclusions of Theorem 1.

Theorem 1 is thus completely proved.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No. 10371065), and the Research Foundation of Doctor Points of China (No. 20060422049). The authors would like to thank the referee for valuable suggestions concerning this paper.

References

[1] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[2] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Kluwer Acad. Publ., Dordrecht, 2003.

[3] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen, Acta Math. **48** (1926), no. 3–4, 367–391.

[4] G. G. Gundersen, Meromorphic functions that share three or four values, J. London Math. Soc. (2) **20** (1979), no. 3, 457–466.

[5] N. Steinmetz, A uniqueness theorem for three meromorphic functions, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), no. 1, 93–110.

[6] M. Reinders, A new example of meromorphic functions sharing four values and a uniqueness theorem, Complex Variables Theory Appl. **18** (1992), no. 3–4, 213–221.

[7] G. G. Gundersen, Meromorphic functions that share four values, Trans. Amer. Math. Soc. **277** (1983), no. 2, 545–567. Correction: **304**, (1987) 847–850.

[8] W. W. Adams and E. G. Straus, Non-archimedian analytic functions taking the same values at the same points, Illinois J. Math. **15** (1971), 418–424.

[9] W. Doeringer, Exceptional values of differential polynomials, Pacific J. Math. **98** (1982), no. 1, 55–62.

[10] G. G. Gundersen, Meromorphic functions that share three values IM and a fourth value CM, Complex Variables Theory Appl. **20** (1992), no. 1–4, 99–106.

[11] E. Mues, Bemerkungen zum vier-Punkte-Satz, in *Complex methods on partial differential equations*, 109–117, Akademie Verlag, Berlin, 1989.

[12] E. Mues, Meromorphic functions sharing four values, Complex Variables Theory Appl. **12** (1989), no. 1–4, 169–179.

[13] E. Mues, Shared value problems for meromorphic functions, in *Value distribution theory and complex differential equations, Joensuu* 1994, 17–43, Univ. Joensuu, Joensuu.

[14] M. Reinders, Eindeutigkeitssätze für meromorphe Funktionen, die vier Werte teilen, Mitt. Math. Sem. Giessen No. 200 (1991), 15–38.

[15] S. P. Wang, On meromorphic functions that share four values, J. Math. Anal. Appl. **173** (1993), no. 2, 359–369.

[16] H. Ueda, Some estimates for meromorphic functions sharing four values, Kodai Math. J. **17** (1994), no. 2, 329–340.

[17] H. Ueda, Meromorphic functions f and g that share two values CM and two other values in the sense of $E_k(\beta, f) = E_k(\beta, g)$, Kodai Math. J. **21** (1998), no. 3, 273–284.

[18] Yi, H. X., and Zhou, C. T.,: Meromorphic functions that share four values. J. Shandong Univ., **31**(1996), 121–128.

[19] T. P. Czubiak and G. G. Gundersen, Meromorphic functions that share pairs of values, Complex Variables Theory Appl. **34** (1997), no. 1–2, 35–46.

[20] G. Qiu, Four values theorem of derivatives of meromorphic functions, Systems Sci. Math. Sci. **11** (1998), no. 3, 245–248.

[21] J.-P. Wang, Meromorphic functions sharing four values, Indian J. Pure Appl. Math. **32** (2001), no. 1, 37–46.

[22] J.-P. Wang, On Nevanlinna's four-value theorem, Southeast Asian Bull. Math. **29** (2005), no. 1, 169–178.

[23] B. Huang, Some results on Gundersen's problem, J. Systems Sci. Math. Sci. **23** (2003), no. 4, 467–481.

[24] B. Huang, On the unicity of meromorphic functions that share four values, Indian J. Pure Appl. Math. **35** (2004), no. 3, 359–372.

[25] B. Huang and J. Du, Meromorphic functions that share four values, Acta Math. Sci. Ser. B Engl. Ed. **24** (2004), no. 4, 529–535.

[26] K. Ishizaki, Uniqueness problems on meromorphic functions which share four small functions, in *Proceedings of the Second ISAAC Congress, Vol.* 1 (*Fukuoka,* 1999), 467–472, Kluwer Acad. Publ., Dordrecht.

[27] P. Li, Meromorphic functions that share four small functions, J. Math. Anal. Appl. **263** (2001), no. 1, 316–326.

[28] W. Yao, Meromorphic functions sharing four small functions on 2CM+2IM, Arch. Math. (Basel), **81** (2003), no. 6, 666–677.