A note on the mean value of the zeta and L-functions. XV

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Abstract: The aim of the present article is to render the spectral theory of mean values of automorphic L-functions – in a unified fashion. This is an outcome of the investigation commenced with the parts XII and XIV, where a framework was laid on the basis of the theory of automorphic representations and a general approach to the mean values was envisaged. We restrict ourselves to the situation offered by the full modular group, solely for the sake of simplicity. Details and extensions are to be published elsewhere.

Key words: Mean values of automorphic L-functions; automorphic representations; Kirillov model.

1. To begin with, we stress that all notations and conventions will stay effective once introduced.

Let $\mathrm{G} = \mathrm{PSL}(2, \mathbf{R})$ and $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. Write

$$\mathrm{n}[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \quad \mathrm{a}[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix},$$

$$\mathrm{k}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and $\mathrm{N} = \{\mathrm{n}[x] : x \in \mathbf{R}\}$, $\mathrm{A} = \{\mathrm{a}[y] : y > 0\}$, $\mathrm{K} = \{\mathrm{k}[\theta] : \theta \in \mathbf{R}/\pi\mathbf{Z}\}$, so that $\mathrm{G} = \mathrm{NAK}$. We read this as $\mathrm{G} \ni \mathrm{g} = \mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta]$; the coordinate (x, y, θ) will retain this definition. The Haar measures on the groups N, A, K, G are normalized, respectively, by $d\mathrm{n} = dx$, $d\mathrm{a} = dy/y$, $d\mathrm{k} = d\theta/\pi$, $d\mathrm{g} = d\mathrm{n}d\mathrm{a}d\mathrm{k}/y$, with Lebesgue measures dx, dy, $d\theta$.

The space $L^2(\Gamma\backslash\mathrm{G})$ is composed of all left Γ-automorphic functions on G, vectors for short, which are square integrable over $\Gamma\backslash\mathrm{G}$ against $d\mathrm{g}$. Elements of G act unitarily on vectors from the right. We have the orthogonal decomposition

$$L^2(\Gamma\backslash\mathrm{G}) = \mathbf{C}\cdot 1 \bigoplus {}^0L^2(\Gamma\backslash\mathrm{G}) \bigoplus {}^eL^2(\Gamma\backslash\mathrm{G})$$

into invariant subspaces. Here ${}^0L^2$ is the cuspidal subspace spanned by vectors whose Fourier expansions with respect to the left action of N have vanishing constant terms. The subspace ${}^eL^2$ is spanned by integrals of Eisenstein series. Invariant subspaces of $L^2(\Gamma\backslash\mathrm{G})$ and Γ-automorphic representations of G are interchangeable concepts.

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The cuspidal subspace decomposes into irreducible subspaces: ${}^0L^2(\Gamma\backslash\mathrm{G}) = \bigoplus V$. The Casimir operator $\Omega = y^2(\partial_x^2 + \partial_y^2) - y\partial_x\partial_\theta$ becomes a constant multiplication in each V so that $\Omega|_{V^\infty} = (\nu_V^2 - \frac14)\cdot 1$, where V^∞ is the set of all infinitely differentiable vectors in V. Under our present supposition, V belongs either to the unitary principal series or to the discrete series; accordingly, we have $\nu_V \in i\mathbf{R}$ or $\nu_V = \ell - \frac12$, $1 \le \ell \in \mathbf{Z}$. The right action of K induces the decomposition of each V into K-irreducible subspaces: $V = \bigoplus_{p=-\infty}^{\infty} V_p$ with $\dim V_p \le 1$. If it is not trivial, V_p is spanned by a Γ-automorphic function on which the right translation by $\mathrm{k}[\theta]$ becomes the multiplication by the factor $\exp(2ip\theta)$. It is called a Γ-automorphic form of spectral parameter ν_V and weight $2p$. If V is in the unitary principal series, then $\dim V_p = 1$ for all $p \in \mathbf{Z}$ and there exists a complete orthonormal system $\{\varphi_p \in V_p : p \in \mathbf{Z}\}$ of V such that

$$\varphi_p(\mathrm{g}, V) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\mathrm{sgn}(n)} \phi_p(\mathrm{a}[|n|]\mathrm{g}; \nu_V),$$

where $\phi_p(\mathrm{g}; \nu) = y^{\nu+\frac12}\exp(2ip\theta)$, and

$$\mathcal{A}^\delta \phi_p(\mathrm{g}; \nu) = \int_{-\infty}^{\infty} \exp(-2\pi i\delta x)\phi_p(\mathrm{wn}[x]\mathrm{g}; \nu)dx,$$

with $\mathrm{w} = \mathrm{k}\left[\frac12\pi\right]$ the Weyl element. Our normalization is such that the coefficients $\varrho_V(n)$ do not depend on the weight. Also we may impose the Hecke invariance; in particular $\varrho_V(-n) = \epsilon_V\varrho_V(n)$ with $\epsilon_V = \pm 1$.

We have skipped the discrete series. In the sequel as well, we shall argue as if there were no discrete series representation. This should not cause any confusion, as the discussions below, especially those pertaining estimations, extend readily to the discrete series. Nevertheless, it should be remarked that the definition of $\varphi_p(\mathrm{g}, V)$ given above has to be modified for those V in the discrete series according to the normalization [2, (2.21) and (2.25)].

2. Our discussion depends much on the uniform bounds for $\mathcal{A}\phi_p(\mathrm{a}[y];\nu)$, $\mathcal{A}=\mathcal{A}^+$, such as [2, (4.3) and (4.5)]. In order to make our argument applicable to any irreducible cuspidal representation, we derive from the latter a bound that is somewhat weaker than the former but is still sufficient for our purpose; in fact the proof of [2, (4.5)] works for all cases. We thus put

$$\Gamma_p(s,\nu)=\int_0^\infty y^{s-1}\mathcal{A}\phi_p(\mathrm{a}[y];\nu)\frac{dy}{y},\quad \operatorname{Re}s>1;$$

we have shifted the argument in [2, (4.10)] by $-\frac{1}{2}$. Let us assume that $\nu\in i\mathbf{R}$. We divide the integral at $y=|p|+|\nu|+1$. To the part with smaller y we apply the fact that $\mathcal{A}^{\mathrm{sgn}(u)}\phi_p(\mathrm{a}[|u|];\nu)$ is a unit vector in $L^2(\mathbf{R}^\times, d^\times/\pi)$, $d^\times u=du/|u|$ (see e.g., [2, Lemma 4]). Hence this part is $\ll(|p|+|\nu|+1)^{\operatorname{Re}s-1}$. On the other hand, by [2, (4.5)] the remaining part is $\ll(|p|+|\nu|+1)^{\operatorname{Re}s-\frac{1}{2}}$. We then invoke the identity

$$\Gamma_p(s,\nu)=4\pi\cdot\frac{\pi\Gamma_p(s+2,\nu)-p\Gamma_p(s+1,\nu)}{(s-\frac{1}{2})^2-\nu^2},$$

which can be proved by applying the operator $\mathcal{D}_\nu=(d/dy)^2-(2\pi)^2-\big(\nu^2-\frac{1}{4}\big)y^{-2}$ either to y^s or to $\mathcal{A}\phi_p(\mathrm{a}[y];\nu)$ in the last integral, on noting that the latter is a constant multiple of the Whittaker function $W_{p,\nu}(4\pi y)$ (see [2, (2.16)]). Then, by Mellin's inversion, we conclude that $\mathcal{A}\phi_p(\mathrm{a}[y];\nu)\ll y^{\frac{1}{2}-\varepsilon}(|p|+|\nu|+1)^{2+\varepsilon}$ for any small $\varepsilon>0$. Hereafter we shall term this and [2, (4.5)] the basic bounds.

The above works with the discrete series as well. Actually, any combination of p,ν such that either $-\frac{1}{2}<\nu<\frac{1}{2}$ or $\nu=\ell-\frac{1}{2}$ with $1\le\ell\in\mathbf{Z}$, $\ell\le|p|$, could also be dealt with, as an explicit evaluation of the norm of $\mathcal{A}^{\mathrm{sgn}(u)}\phi_p(\mathrm{a}[|u|];\nu)$ in $L^2(\mathbf{R}^\times, d^\times/\pi)$ can then be performed by using [2, (4.14)] that is proved in [5] (see also [2, pp. 98–99]). Further extensions to the situation $\nu\notin i\mathbf{R}$ can be obtained by iterating the last recursive relation amongst the values of Γ_p. We remark that $\mathcal{D}_\nu$ is connected with the operator ∂_θ via the Kirillov map.

3. We now define the automorphic L-function associated with an irreducible representation V by

$$L_V(s)=\sum_{n=1}^\infty\varrho_V(n)n^{-s}.$$

This converges absolutely for $\operatorname{Re}s>1$ and continues to an entire function which is of a polynomial order both in s and in ν_V if $\operatorname{Re}s$ is bounded.

We fix an A among V's, and consider the mean square

$$\mathcal{M}(A;g)=\int_{-\infty}^\infty\big|L\big(\tfrac{1}{2}+it\big)\big|^2g(t)dt,\quad L=L_A,$$

where the weight function g is assumed, for the sake of simplicity, to be even, entire, real on $\mathbf{R}$, and of fast decay in any fixed horizontal strip. Our aim is to establish a full spectral decomposition of $\mathcal{M}(A;g)$. Our method is applicable to any representation, though we shall deal with only the case where A is in the unitary principal series. The fourth moment of the Riemann zeta-function could be discussed equally.

We start with the integral

$$I(u,v;g)=\int_{-\infty}^\infty\overline{L(\bar u+it)}L(v+it)g(t)dt.$$

This is entire in u, v; and $\mathcal{M}(A;g)=I\big(\frac{1}{2},\frac{1}{2};g\big)$. In the region of absolute convergence, we have

$$I(u,v;g)=\frac{R(u+v)}{\zeta(2(u+v))}\hat g(0)+J(u,v;g)+\overline{J(\bar v,\bar u;g)}.$$

Here R is the Rankin zeta-function attached to A, $\hat g$ the Fourier transform of g, and

$$J(u,v;g)=\sum_{f=1}^\infty\sum_{n=1}^\infty\frac{\overline{\varrho(n)}\varrho(n+f)}{(2n+f)^{u+v}}\left(\frac{\sqrt{n(n+f)}}{2n+f}\right)^{2\alpha}\cdot\tilde g(f/(2n+f);u,v),$$

where $\varrho=\varrho_A$ and

$$\tilde g(x;u,v)=2^{u+v+2\alpha}\frac{\hat g(\log((1+x)/(1-x)))}{(1-x)^{u+\alpha}(1+x)^{v+\alpha}},$$

with $0\le x\le 1$. Here α is a sufficiently large positive integer, which is implicit throughout the sequel. We have slightly modified our procedure given in [6].

Let g^* be the Mellin transform of $\tilde g$. It is immediate to see that $g^*(s;u,v)$ is of rapid decay with respect to s, provided $\operatorname{Re}s$ and u,v are bounded; moreover, $g^*(s;u,v)/\Gamma(s)$ is entire over $\mathbf{C}^3$. Thus, by Mellin's inversion,

$$J(u,v;g)$$
$$=\frac{1}{2\pi i}\int_{(\eta)}\left\{\sum_{f=1}^{\infty}f^{-s}D_f(u+v-s)\right\}g^*(s;u,v)ds,$$

where (η) is the line $\mathrm{Re}\, s=\eta>0$, and

$$D_f(s)=\sum_{n=1}^{\infty}\frac{\overline{\varrho(n)}\varrho(n+f)}{(2n+f)^s}\left(\frac{\sqrt{n(n+f)}}{2n+f}\right)^{2\alpha}.$$

Checking the convergence, we see that the condition $\mathrm{Re}\,(u+v)>\max\{2,1+\eta\}$ is required here.

4. Now, we need to have a full spectral decomposition of $D_f(s)$ which is to yield a continuation, with a polynomial growth, to the left of $\mathrm{Re}\, s=1$ so that $J(u,v;g)$ can also be continued to a neighbourhood of the point $\left(\frac12,\frac12\right)$. Recently, V. Blomer and G. Harcos [1] succeeded in establishing such an assertion on $D_f(s)$. They started with an old idea of ours, published in [6], to employ the Kirillov model to pick up a favourable vector in dealing with the problem of the spectral decomposition of $D_f(s)$, and proceeded one step further by a use of the Sobolev norms with which they could derive the crucial polynomial growth that we had left open in [6]. In what follows, we shall take an alternative way by modifying [6] with certain simple devices from [2] and [5]. The present article can be read independently of [1].

Thus, let A, α be as above, and τ a parameter in the right half plane; all implicit constants in the sequel may depend on A, α and $\mathrm{Re}\,\tau$ at most. We apply the inverse Kirillov map to the function $w(y,\tau)$ which is equal to $y^{\alpha+\frac12}\exp(-\tau y)$ for $y>0$, and vanishes for $y\le 0$. By [2, Lemma 4] there exists a vector $\Phi(\mathrm{g},\tau)$ in A such that

$$\Phi(\mathrm{n}[x]\mathrm{a}[y],\tau)=\sum_{n=1}^{\infty}\frac{\varrho(n)}{\sqrt{n}}w(ny,\tau)\exp(2\pi i n x).$$

More precisely, it equals $\sum_{p=-\infty}^{\infty}a_p(\tau)\varphi_p(\mathrm{g},A)$, with

$$a_p(\tau)=\frac{1}{\pi}\int_0^{\infty}w_\alpha(y,\tau)\overline{\mathcal{A}\phi_p(\mathrm{a}[y];\nu_A)}\,\frac{dy}{y}.$$

The function $\Phi(\mathrm{g},\tau)$ is regular for $\mathrm{Re}\,\tau>0$, since

$$a_p(\tau)\ll(|\tau|+1)^{2\alpha}(|p|+1)^{-\alpha}$$

and $\varphi_p(\mathrm{g},A)\ll(|p|+1)^2$, for any g, as can be shown by [2, (4.5)].

To prove this bound for $a_p(\tau)$, we use the operator $\mathcal{D}_\nu$ again: We may assume that $p\ne 0$; then,

$$a_p(\tau)=-\frac{1}{4\pi p}\int_0^{\infty}w_\alpha(y,\tau)\overline{\mathcal{D}_{\nu_A}\mathcal{A}\phi_p(\mathrm{a}[y];\nu_A)}\,dy,$$

Integrate in parts, and repeat the procedure α times; and we get the bound.

5. Next, we put $\Psi(\mathrm{g},\tau)=\Phi(\mathrm{g},\tau)\overline{\Phi(\mathrm{g},\overline{\tau})}$. We have, for any integer $f\ge 0$,

$$\int_0^1\Psi(\mathrm{n}[x]\mathrm{a}[y],\tau)\exp(-2\pi i f x)dx$$
$$=y^{2\alpha+1}\sum_{n=1}^{\infty}\overline{\varrho(n)}\varrho(n+f)(n(n+f))^{\alpha}$$
$$\cdot\exp(-(2n+f)\tau y).$$

The Parseval formula implies that the left side is equal to

$$\sum_V\frac{\varrho_V(f)}{\sqrt{f}}\Delta(fy,\tau;V)$$
$$+\int_{(0)}\frac{f^{-\nu}\sigma_{2\nu}(f)}{\sqrt{f}\zeta(1+2\nu)}\Delta(fy,\tau;\nu)\frac{d\nu}{4\pi i},$$

where V runs over all irreducible cuspidal representations, and

$$\Delta(y,\tau;V)=\sum_{p=-\infty}^{\infty}\langle\Psi(\cdot,\tau),\varphi_p(\cdot,V)\rangle\mathcal{A}\phi_p(\mathrm{a}[y];\nu_V),$$
$$\Delta(y,\tau;\nu)=\sum_{p=-\infty}^{\infty}\langle\Psi(\cdot,\tau),E_p(\cdot,\nu)\rangle\mathcal{A}\phi_p(\mathrm{a}[y];\nu).$$

Here $\langle\,,\rangle$ is the natural inner product on $L^2(\mathit{\Gamma}\backslash\mathrm{G})$, and E_p is the Eisenstein series of weight $2p$ (see [2, (3.19)]).

The convergence of the spectral expansion of Ψ is in fact absolute and fast, provided α is sufficiently large. To confirm this, we apply the operator $\Omega+i\partial_\theta^2$ repeatedly as is done at [2, (5.9)]; and we get, via the above bound for $a_p(\tau)$,

$$\langle\Psi(\cdot,\tau),\varphi_p(\cdot,V)\rangle\ll(|\tau|+1)^{4\alpha}(|\nu_V|+|p|)^{-\frac12\alpha},$$

Then we appeal to the basic bounds, getting

$$\Delta(y,\tau;V)$$
$$\ll(|\tau|+1)^{4\alpha}(|\nu_V|+1)^{-\frac14\alpha}y^{\frac12-\varepsilon}(1+y)^{-\frac15\alpha}.$$

The contribution of the continuous spectrum or rather the function $\Delta(y,\tau;\nu)$ is to be discussed later. This yields our assertion.

In order to pick up a particular Fourier coefficient, we have used the projection procedure as was done in [1] to avoid an inner product argument that had been applied in [6]. It is worth pointing it out that in [2, Sections 3–5] a procedure of the same kind was developed, employing the Kirillov model as a main implement to evaluate projections to irreducible subspaces of a certain Poincaré series, explicitly in terms of its seed function. Therefore, we surmise that the projection $\Delta(y,\tau;\cdot)$ could also be handled more precisely with a modification of the argument of [2]. To this issue we shall return elsewhere. In passing, we remark that $\Delta(y,\tau;\cdot)$ is regular for $\operatorname{Re}\tau>0$.

6. We are now to make the last estimation procedure explicit. Thus, we note that

$$\begin{aligned}&\big|(\overline{\nu_V}^2-\tfrac14-i(2p)^2)^q\langle\Psi(\cdot,\tau),\varphi_p(\cdot,V)\rangle\big|\\ &=\big|\langle(\Psi(\cdot,\tau),(\Omega-i\partial_\theta^2)^q\varphi_p(\cdot,V)\rangle\big|\\ &=\big|\langle(\Omega+i\partial_\theta^2)^q\Psi(\cdot,\tau),\varphi_p(\cdot,V)\rangle\big|\\ &\le\big\|(\Omega+i\partial_\theta^2)^q\Psi(\cdot,\tau)\big\|,\end{aligned}$$

for any integer $q\ge0$. By definition

$$\Psi(\mathrm{g},\tau)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}a_k(\tau)\overline{a_l(\bar\tau)}\varphi_k(\mathrm{g})\overline{\varphi_l(\mathrm{g})}$$

with $\varphi_k(\mathrm{g})=\varphi_k(\mathrm{g},A)$, $\varphi_l(\mathrm{g})=\varphi_l(\mathrm{g},A)$. Since

$$\Omega=\tfrac14\mathrm{E}\overline{\mathrm{E}}-\tfrac14\partial_\theta^2+\tfrac12 i\partial_\theta,$$

with the Maass operator $\mathrm{E}=e^{2i\theta}(2iy\partial_x+2y\partial_y-i\partial_\theta)$, we see that $\Omega\varphi_k\overline{\varphi_l}$ is a linear combination of $\varphi_{k+j}\overline{\varphi_{l+j}}$, $j=-1,0,1$, the coefficients of which are polynomials of the second degree on k,l; note that ν_A is now regarded as a constant. Thus

$$(\Omega+i\partial_\theta^2)^q\Psi(\mathrm{g},\tau)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}b_{k,l}(\tau;q)\varphi_k(\mathrm{g})\overline{\varphi_l(\mathrm{g})},$$

where

$$b_{k,l}(\tau;q)=\sum_{j=-q}^{q}d_j(k,l)a_{k+j}(\tau)\overline{a_{l+j}(\bar\tau)}$$

with a polynomial $d_j(k,l)$ of degree $2q$ on k,l. We have

$$b_{k,l}(\tau;q)\ll(|k|+|l|+1)^{2q}\frac{(|\tau|+1)^{4\alpha}}{((|k|+1)(|l|+1))^{\alpha}},$$

for each fixed q. We then note as before that $\varphi_k(\mathrm{g})\ll(|k|+1)^2$; and thus, with $q=\left[\frac13\alpha\right]$, say, we get the uniform bound $(\Omega+i\partial_\theta^2)^q\Psi(\mathrm{g},\tau)\ll(|\tau|+1)^{4\alpha}$, which gives the first inequality in the last section.

7. As to the continuous spectrum, we invoke the functional equation for E_p ([2, (3.33)]), and have, for $\operatorname{Re}\nu=0$,

$$\begin{aligned}&\langle(\Omega+i\partial_\theta^2)^q\Psi(\cdot,\tau),E_p(\cdot,\nu)\rangle\\ &=\gamma_p(\nu)\int_{\Gamma\backslash\mathrm{G}}(\Omega+i\partial_\theta^2)^q\Psi(\mathrm{g},\tau)E_{-p}(\mathrm{g},\nu)d\mathrm{g},\end{aligned}$$

with

$$\gamma_p(\nu)=\pi^{-2\nu}\frac{\zeta(1+2\nu)}{\zeta(1-2\nu)}\frac{\Gamma\big(\frac12+\nu+p\big)}{\Gamma\big(\frac12-\nu+p\big)}.$$

Assuming $\operatorname{Re}\nu>\frac12$, we unfold the last integral, and see that it equals $R\big(\nu+\frac12\big)Y_p(\nu,\tau;q)/\zeta(2\nu+1)$, where R is as above, and

$$\begin{aligned}&Y_p(\nu,\tau;q)=\sum_{l=-\infty}^{\infty}b_{l+p,l}(\tau;q)\\ &\cdot\sum_{\delta=\pm}\int_0^\infty\mathcal{A}^\delta\phi_{l+p}(\mathrm{a}[y],\nu_A)\overline{\mathcal{A}^\delta\phi_l(\mathrm{a}[y],\nu_A)}y^{\nu-\frac32}dy.\end{aligned}$$

Again by the basic bounds, we see that $Y_p(\nu,\tau;q)$ is regular and $\ll(|\tau|+1)^{4\alpha}$ for $\operatorname{Re}\tau>0$ and $\operatorname{Re}\nu>-\frac12$. Hence, in the same domain,

$$\Delta(y,\tau;\nu)=\frac{R\big(\nu+\frac12\big)}{\zeta(1-2\nu)}\Lambda(y,\tau;\nu),$$

with

$$\begin{aligned}&\Lambda(y,\tau;\nu)=\\ &\sum_{p=-\infty}^{\infty}\frac{\pi^{-2\nu}Y_p(\nu,\tau;q)}{\big(\nu^2-\frac14-i(2p)^2\big)^q}\frac{\Gamma\big(\frac12+\nu+p\big)}{\Gamma\big(\frac12-\nu+p\big)}\mathcal{A}\phi_p(\mathrm{a}[y];\nu).\end{aligned}$$

One may conclude, via [2, (4.3) and (4.5)], that

$$\begin{aligned}&\Lambda(y,\tau;\nu)\\ &\ll(|\tau|+1)^{4\alpha}(|\nu|+1)^{-\frac14\alpha}y^{\frac12-|\operatorname{Re}\nu|-\varepsilon}(1+y)^{-\frac15\alpha}\end{aligned}$$

for $\operatorname{Re}\tau>0$ and $\operatorname{Re}\nu>-\frac12$.

8. We now set $\tau=s$, and observe that

$$\begin{aligned}D_f(s)&=\frac{s^{s+2\alpha}}{\Gamma(s+2\alpha)}\int_0^\infty y^{s-2}\\ &\quad\cdot\int_0^1\Psi(\mathrm{n}[x]\mathrm{a}[y],s)\exp(-2\pi ifx)dxdy\\ &=\sum_V f^{\frac12-s}\varrho_V(f)\Xi(s,V)\\ &\quad+\int_{(0)}\frac{f^{\frac12-\nu}\sigma_{2\nu}(f)}{\zeta(1+2\nu)\zeta(1-2\nu)}R\big(\nu+\tfrac12\big)\,\Xi(s,\nu)\frac{d\nu}{4\pi i},\end{aligned}$$

where

$$\Xi(s,V)=\frac{s^{s+2\alpha}}{\Gamma(s+2\alpha)}\int_0^\infty y^{s-2}\Delta(y,s;V)dy,$$
$$\Xi(s,\nu)=\frac{s^{s+2\alpha}}{\Gamma(s+2\alpha)}\int_0^\infty y^{s-2}\Lambda(y,s;\nu)dy.$$

The bound for $\Delta(y,\tau;V)$ implies that $\Xi(s,V)$ is regular and $\ll |s|^{4\alpha+\frac12}(|\nu_V|+1)^{-\frac14\alpha}$ for $\mathrm{Re}\, s>\frac12$. Similarly, $\Xi(s,\nu)$ is regular and $\ll |s|^{4\alpha+\frac12}(|\nu|+1)^{-\frac14\alpha}$ for $\mathrm{Re}\, s>\frac12+|\mathrm{Re}\,\nu|$ and $\mathrm{Re}\,\nu>-\frac12$.

Therefore we have proved that $D_f(s)$ is indeed regular and of polynomial growth for $\mathrm{Re}\, s>\frac12$ ([1, Theorem 2]).

It is to be observed that in order to offset the exponential growth of the factor $1/\Gamma(s+2\alpha)$ we have adopted an old idea of Ju.V. Linnik which he introduced in his investigation on approximate functional equations for Dirichlet L-functions.

9. We return to the function $J(u,v;g)$; thus we impose $\mathrm{Re}\,(u+v)>\max\{2,1+\eta\}$ initially. In view of the fast decay of $g^*(s;u,v)$, the last assertion on $D_f(s)$ yields immediately that

$$J(u,v;g)=\sum_V L_V\big(u+v-\tfrac12\big)\,\Theta(u,v;V;g)$$
$$+\frac{1}{4\pi i}\int_{(0)}\frac{\zeta(u+v-\frac12+\nu)\zeta(u+v-\frac12-\nu)}{\zeta(1+2\nu)\zeta(1-2\nu)}\cdot R\big(\tfrac12+\nu\big)\,\Theta(u,v;\nu;g)d\nu,$$

where

$$\Theta(u,v;V;g)=\frac{1}{2\pi i}\int_{(\eta)}\Xi(u+v-\xi,V)g^*(\xi;u,v)d\xi,$$

and the factor $\Xi(u+v-\xi,\nu)$ appears instead in the continuous spectrum. The Θ is of fast decay either in ν_V or in ν, as Ξ is.

We fix a sufficiently small $\varepsilon>0$; we may move the last contour to (ε), provided $\mathrm{Re}\,(u+v)>\frac23$, say. Hence, the last expansion of $J(u,v;g)$ holds under $\mathrm{Re}\,(u+v)>\frac32$. This lower bound is required to get the factors $L_V\big(u+v-\frac12\big)$ and $\zeta\big(u+v-\frac12\pm\nu\big)$. However, the former is entire and of a polynomial order in ν_V if u,v are bounded. Thus the cuspidal part of $J(u,v;g)$ is regular in a neighbourhood of the point $\big(\frac12,\frac12\big)$ at which it takes the value

$$\sum_V L_V\big(\tfrac12\big)\,\Theta_A(V;g),$$

with $\Theta_A(V;g)=\Theta\big(\frac12,\frac12;V;g\big)$.

As we are about to deal with the continuous spectrum, we should remark that $\Theta(u,v;\nu;g)$ remains regular in the three complex variables and of fast decay in ν, throughout the procedure below, because of the property of $\Xi(s,\nu)$ mentioned above. Thus, we restrict (u,v) so that $2>\mathrm{Re}\,(u+v)>\frac32$. Then, in the last expression for $J(u,v;g)$ one may shift the ν-contour to $\big(\frac12+\varepsilon\big)$, encountering the pole at $u+v-\frac32$ with the residue

$$-\frac{R(u+v-1)}{\zeta(2(2-u-v))}\Theta\big(u,v;u+v-\tfrac32;g\big)$$

as well as those of the factor $R(\frac12+\nu)/\zeta(1-2\nu)$; we may assume, without loss of any generality, that u,v are such that all the residues in question are finite. This yields a meromorphic continuation of the continuous spectrum part, so that one may move (u,v) close to $\big(\frac12,\frac12\big)$ as far as $\mathrm{Re}\,(u+v)>1$ is satisfied; this condition is needed to have the last Θ factor defined well. Then, shift the ν-contour back to the original. All the residual contribution coming from $R(\frac12+\nu)/\zeta(1-2\nu)$ cancel out those arising from the previous shift of the contour. Only the pole at $\frac32-u-v$ contributes newly. The resulting integral is regular at $\big(\frac12,\frac12\big)$; we get the factor $\Theta_A(\nu;g)=\Theta\big(\frac12,\frac12;\nu;g\big)$, $\mathrm{Re}\,\nu=0$.

10. Collecting all the above, we obtain

Theorem. *We have the spectral decomposition*

$$\mathcal{M}(A;g)=m(A;g)+2\mathrm{Re}\left\{\sum_V L_V\big(\tfrac12\big)\,\Theta_A(V;g)\right.$$
$$\left.+\int_{(0)}\frac{\zeta(\frac12+\nu)\zeta(\frac12-\nu)}{\zeta(1+2\nu)\zeta(1-2\nu)}R_A\big(\tfrac12+\nu\big)\Theta_A(\nu;g)\frac{d\nu}{4\pi i}\right\},$$

where $R_A=R$, and $m(A;g)$ is the value at $\big(\frac12,\frac12\big)$ of the function

$$\frac{R(u+v)}{\zeta(2(u+v))}\hat g(0)+\frac{1}{\zeta(2(2-u-v))}$$
$$\cdot\Big\{R(u+v-1)\Theta\big(u,v;u+v-\tfrac32;g\big)$$
$$+R(1-u-v)\Theta\big(u,v;\tfrac32-u-v;g\big)$$
$$+R(u+v-1)\Theta\big(v,u;u+v-\tfrac32;g\big)$$
$$+R(1-u-v)\Theta\big(v,u;\tfrac32-u-v;g\big)\Big\}.$$

Albeit the Θ factors in the last expression is defined so far only under the condition $2>\mathrm{Re}\,(u+v)>1$, the expression can in fact be continued to a neighbourhood of $\big(\frac12,\frac12\big)$, for $I(u,v;g)$ and all other parts in the spectral expansion of $J(u,v;g)$ and $\overline{J(\bar v,\bar u;g)}$ are regular there. We could make the continuation procedure more explicit, using the property of Γ_p given in the second section, but the above suffices for our present aim.

Our theorem is to be compared with [3, Theorem] and [4, Theorem 4.2] which respectively deal with the mean square of automorphic L-functions associated with discrete series representations and with that of the product of two values of the Riemann zeta-function, i.e., the L-function associated with the continuous spectrum or Eisenstein series. Unlike those highly explicit results, admittedly the above asserts only the existence of a full spectral decomposition for $\mathcal{M}(A; g)$. The construction of the transform $\Theta_A(\cdot; g)$ has to be made explicit in terms of the weight function g before one attempts any application; we shall take this task in our forthcoming works. Nevertheless, the present work could be a means to broaden the perspective of the theory of the mean values of the zeta and L-functions that is rendered in [7] via particular examples.

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