On the totally ramified value number of the Gauss map of minimal surfaces

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Abstract: In this paper, we define the totally ramified value number (TRVN) of the Gauss map of a complete minimal surface, and prove that there exist algebraic minimal surfaces that have the TRVN equal to 2.5.

Key words: Minimal surface; Gauss map; the totally ramified value number (TRVN).

1. Introduction. The study of the Gauss map of a complete minimal surface in $\mathbf{R}^3$ has achieved many important results and also given rise to many problems. Among them a very interesting problem is the following: if $x\colon M \to \mathbf{R}^3$ is a non-flat complete regular minimal surface, and $g\colon M \to \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is its Gauss map, how many values can g miss? In 1961 Osserman [7] proved that g can miss at most a subset of points of logarithmic capacity 0. In 1981 Xavier [8] proved that g can miss at most 6 values. In 1988 Fujimoto [1] finally proved by value distribution theoretical method that g can miss at most 4 values. This is the best possible upper bound since a lot of examples of complete minimal surfaces whose Gauss maps miss 4 values exist (for example, the classical Scherk surface). Moreover, Fujimoto investigated the totally ramified value number, which gives more detailed information than the number of exceptional values.

Definition 1.1 ([5])**.** Let M be a Riemann surface and f a meromorphic function on M. We call $b \in \hat{\mathbf{C}}$ *a totally ramified value* of f when at all the inverse image points of b, f branches. We regard exceptional values also as totally ramified values. Let $\{a_1, \dots, a_r, b_1, \dots, b_l\} \in \hat{\mathbf{C}}$ be the set of all totally ramified values of f, where a_j $(j = 1, \dots, r)$ are exceptional values. For each a_j, put $\nu_j = \infty$, and for each b_j, define ν_j to be the minimum of the multiplicities of f at points $f^{-1}(b_j)$. Then we have $\nu_j \geq 2$. We call

$$(1) \qquad \nu_f = \sum_{a_j, b_j} \left(1 - \frac{1}{\nu_j}\right) = r + \sum_{j=1}^{l} \left(1 - \frac{1}{\nu_j}\right)$$

the totally ramified value number (TRVN) of f.

Since the Gauss map of a minimal surface in $\mathbf{R}^3$ is a meromorphic function on an open Riemann surface, we can define its TRVN. Fujimoto proved the following theorem.

Theorem 1.1 ([2])**.** *Let $x\colon M \to \mathbf{R}^3$ be a non-flat complete regular minimal surface, g be its Gauss map. Let D_g be the number of exceptional values of g and ν_g be the TRVN of g. Then we have*

$$D_g \leq \nu_g \leq 4.$$

On the other hand, in 1964 Osserman proved the following theorem.

Theorem 1.2 ([6, 7])**.** *Let $x\colon M \to \mathbf{R}^3$ be a non-flat algebraic minimal surface, by which we mean a complete regular minimal surface with finite total curvature. Then we have*

$$D_g \leq 3.$$

But there is no known example with $D_g = 3$. Since the Catenoid is an example with $D_g = 2$, either "2" or "3" is the best possible upper bound for D_g. Since Osserman posed this problem in 1967, it has not been solved yet. Many people believe "2" is the best possible upper bound. It has also been believed that ν_g is bounded from above by 2. In this paper, we show that there exist algebraic minimal surfaces that have $\nu_g = 2.5$, i.e., ν_g is strictly larger than 2. It is hoped that Osserman's problem will be solved by using Nevanlinna theory or value distribution theoretical method. Therefore the existence of examples of algebraic minimal sufaces with $\nu_g = 2.5$ implies that we must find a new "geometry" which is not presented in arguments in [1, 2]. The

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forthcoming paper [3] is our first attempt toward this direction.

2. Results. We start with the Enneper-Weierstrass representation for a minimal surface in $\mathbf{R}^3$.

Theorem 2.1 ([7]). *Let M be an open Riemann surface, $\omega = h\,dz$ a non-zero holomorphic 1-form and g a non-constant meromorphic function on M. Assume that the poles of g of order k coincide exactly with the zeros of ω of order $2k$ and that the holomorphic 1-forms ϕ_1, ϕ_2, ϕ_3 defined by*

$$\phi_1 = \frac{1}{2}\left(1 - g^2\right)\omega,$$
$$\phi_2 = \frac{\sqrt{-1}}{2}\left(1 + g^2\right)\omega,$$
$$\phi_3 = g\omega$$

have no real periods, i.e., for any cycle $\gamma \in H_1(M, \mathbf{Z})$, $\operatorname{Re}\int_\gamma \phi_j = 0$ $(j = 1, 2, 3)$. Then the surface defined by

$$x = \operatorname{Re}\int(\phi_1, \phi_2, \phi_3)\colon M \to \mathbf{R}^3$$

is a minimal surface immersed in $\mathbf{R}^3$ whose Gauss map is g. The induced metric is given by

$$ds^2 = \frac{1}{4}(1 + |g|^2)^2|\omega|^2.$$

We call the above (ω, g) the *Weierstrass data.* The Gauss curvature K of (M, ds^2) is given by

$$K = -\left(\frac{4|g'|}{|h|(1 + |g|^2)^2}\right)^2,$$

and the total curvature by

$$\tau(M) = \int_M K\,dA = -\int_M \left(\frac{2|g'|}{1 + |g|^2}\right)^2 du\,dv$$

where $z = u + \sqrt{-1}v$ and dA is the surface element of (M, ds^2). Thus $|\tau(M)|$ is the area of M with respect to the metric induced from the Fubini-Study metric of $\hat{\mathbf{C}} = P^1(\mathbf{C})$ by g.

Next, we prove that there exist algebraic minimal surfaces with $\nu_g = 2.5$.

Main theorem. *There exist algebraic minimal surfaces with $\nu_g = 2.5$. In fact, the Weierstrass data of these algebraic minimal surfaces are defined on $M = \hat{\mathbf{C}} \setminus \{\pm\sqrt{-1}, \infty\}$, and they are explicitly given by*

$$(2)\quad (\omega, g) = \left(\frac{(z^2 + t)^2}{(z^2 + 1)^2}\,dz, \sigma\frac{z^2 + 1 + a(t - 1)}{z^2 + t}\right),$$

where a, t are real numbers such that $(a - 1)(t - 1) \neq 0$ and $\sigma^2 = (t + 3)/(a\{(t - 1)a + 4\}) < 0$.

Proof. In [4], it is shown that the surfaces defined by (2) are algebraic minimal surfaces. Here, we show that they have $\nu_g = 2.5$. g misses σa at $z = \pm\sqrt{-1}$ and σ at $z = \infty$. Thus g misses two values. Since g branches at $z = 0$ and the degree of g is 2, $\sigma(1 + a(t - 1))/t$ $(= g(0))$ is a totally ramified value. Therefore, by (1), $\nu_g = 1 + 1 + 1/2 = 5/2$ $(= 2.5)$. □

Finally, we mention the result of [3]. In [3], we obtain an estimate for ν_g.

Theorem 2.2 ([3]). *Let $x\colon M \to \mathbf{R}^3$ (where M is a punctured Riemann surface obtained from a compact Riemann surface of genus G removing k points) be an algebraic minimal surface and d the degree of g. Then we have*

$$(3)\quad \nu_g \leq 2 + \frac{2}{R}, \quad R = \frac{d}{G - 1 + k/2} > 1.$$

In particular, we have $\nu_g < 4$.

The examples in Main theorem show that (3) is the best possible for the case of $(G, k, d) = (0, 3, 2)$. But we do not know whether an algebraic minimal surface with $\nu_g > 2.5$ exists.

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