Global generic Bernstein-Sato polynomial on an irreducible affine scheme

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Abstract: Given p polynomials with coefficients in a commutative unitary integral ring $\mathcal{C}$ containing $\mathbf{Q}$, we define the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme $V \subset \mathrm{Spec}(\mathcal{C})$. We prove the existence of such a non zero rational polynomial which covers and generalizes previous existing results by H. Biosca. When $\mathcal{C}$ is the ring of an algebraic or analytic space, we deduce a stratification of the space of the parameters such that on each stratum, there is a non zero rational polynomial which is a Bernstein-Sato polynomial for any point of the stratum. This generalizes a result of A. Leykin obtained in the case $p = 1$.

Key words: Generic Bernstein-Sato polynomial; Bernstein-Sato polynomial.

Introduction and main results. Fix $n \geq 1$ and $p \geq 1$ two integers and $v \in \mathbf{N}^p$. Let $x = (x_1, \ldots, x_n)$ and $s = (s_1, \ldots, s_p)$ be two systems of variables. Let $\mathbf{k}$ be a field of characteristic 0.**) Let $\mathbf{A}_n(\mathbf{k})$ be the ring of differential operators with coefficients in $\mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$ and $\mathcal{D}$ (resp. $\mathcal{O}$) be the sheaf of rings of differential operators (resp. analytic functions) on $\mathbf{C}^n$ for which we denote by $\mathcal{D}_{x_0}$ (resp. $\mathcal{O}_{x_0}$) the fiber in x_0.

Let $f = (f_1, \ldots, f_p)$ be in $\mathbf{k}[x]^p$ (resp. $\mathcal{O}_{x_0}^p$) and consider the following functional identity:

$$b(s)f^s \in \mathbf{A}_n(\mathbf{k})[s] \cdot f^{s+v},$$

(resp. $\mathcal{D}_{x_0}[s]$ instead of $\mathbf{A}_n(\mathbf{k})[s]$) where $f^{s+v} = f_1^{s_1+v_1} \cdots f_p^{s_p+v_p}$. This identity takes place in the free module generated by f^s over $\mathbf{k}[x, 1/(f_1 \cdots f_p), s]$ (resp. $\mathcal{O}_{x_0}[1/(f_1 \cdots f_p, s)]$).

The set of such $b(s)$ is an ideal of $\mathbf{k}[s]$ (resp. $\mathbf{C}[s]$). This ideal is called the (global) Bernstein-Sato ideal of f (resp. local Bernstein-Sato ideal in x_0) and we denote it by $\mathcal{B}^v(f)$ (resp. $\mathcal{B}^v_{x_0}(f)$). When $p = 1$, this ideal is principal and its monic generator is called the Bernstein polynomial associated with f. Historically, I. N. Bernstein [Be] introduced the (global) Bernstein polynomial and proved its existence (i.e. the fact that it is not zero). J. E. Björk [Bj] has given the proof in the analytic case. Let us cite also M. Kashiwara [Ka] who proved, moreover, the rationality of the roots of the local Bernstein polynomial. For $p \geq 2$, the algebraic case can be easily treated in the same way as for $p = 1$. For the analytic case, the proof of the non nullity of $\mathcal{B}^v_{x_0}(f)$ is due to C. Sabbah ([Sa1, Sa2]). Let us also cite A. Gyoja [Gy] who proved that $\mathcal{B}^v_{x_0}(f)$ contains a non zero *rational* polynomial. The absolute Bernstein-Sato polynomial naturally leads to the notion of a generic Bernstein-Sato polynomial which we shall explain in what follows.

Let $\mathcal{C}$ be a unitary commutative integral ring with the following condition:

For any prime ideal $\mathcal{P} \subset \mathcal{C}$ and for any $n \in \mathbf{N} \setminus \{0\}$, we have:

$$n \in \mathcal{P} \Rightarrow 1 \in \mathcal{P}.$$

This condition is equivalent to the fact that for any $\mathcal{P} \subset \mathcal{C}$, the fraction field of $\mathcal{C}/\mathcal{P}$ is of characteristic 0. Note that this condition is satisfied if and only if there exists an injective ring morphism $\mathbf{Q} \hookrightarrow \mathcal{C}$.

We shall see $\mathcal{C}$ as the ring of coefficients or parameters. Indeed, let $f = (f_1, \ldots, f_p)$ in $\mathcal{C}[x]^p = \mathcal{C}[x_1, \ldots, x_n]^p$.

Let us denote by $\mathbf{A}_n(\mathcal{C})$ the ring of differential operators with coefficients in $\mathcal{C}[x]$, that is the $\mathcal{C}$-algebra generated by x_i and ∂_{x_i} $(i = 1, \ldots, n)$

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**) All the fields considered in this paper are of characteristic 0.

where the only non trivial commutation relations are $[\partial_{x_i}, x_i] = 1$ for $i = 1, \ldots, n$ (hence $\mathcal{C}$ is in the center of $\mathbf{A}_n(\mathcal{C})$).

We denote by $\mathrm{Spec}(\mathcal{C})$ (resp. $\mathrm{Specm}(\mathcal{C})$) the set of prime (resp. maximal) ideals of $\mathcal{C}$ which is the spectrum of $\mathcal{C}$ (resp. the maximal spectrum). For an ideal $\mathcal{I} \subset \mathcal{C}$, we denote by $V(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C}); \mathcal{P} \supset \mathcal{I}\}$ the affine scheme defined by $\mathcal{I}$ and $V_m(\mathcal{I}) = V(\mathcal{I}) \cap \mathrm{Specm}(\mathcal{C})$. Remark that we shall only work with the closed subsets of $\mathrm{Spec}(\mathcal{C})$ and forget the sheaf structure of a scheme.

We are going to introduce the notion of a generic Bernstein-Sato polynomial of f on an irreducible affine scheme $V = V(\mathcal{Q}) \subset \mathrm{Spec}(\mathcal{C})$ (that is when $\mathcal{Q}$ is prime).

So let $\mathcal{Q}$ be a prime ideal of $\mathcal{C}$ and suppose that none of the f_j's is in $\mathcal{Q}[x]$.

The main result of this article is the following.

Theorem 1. *There exists* $h \in \mathcal{C} \setminus \mathcal{Q}$ *and* $b(s) \in \mathbf{Q}[s_1, \ldots, s_p] \setminus 0$ *such that*

$$h\, b(s) f^s \in \mathbf{A}_n(\mathcal{C})[s] f^{s+v} + \left(\mathcal{Q}\Big[x, \frac{1}{f_1 \cdots f_p}, s\Big]\right) f^s.$$

Such a $b(s)$ is called a (rational) generic Bernstein-Sato polynomial of f on $V = V(\mathcal{Q})$ (see the notation and the remark below).

In the case where $p = 1$, the generic and relative (not introduced here) Bernstein polynomial has been studied by F. Geandier in [Ge] and by J. Briançon, F. Geandier and P. Maisonobe in [Br-Ge-Ma] in an analytic context (where f is an analytic function of x). In [Bi] (see also [Bi2]), H. Biosca studied these notions with $p \geq 1$ in the analytic and the algebraic context (that which we are concerned with) and proved that when

- $\mathcal{C} = \mathbf{C}[a_1, \ldots, a_m]$ or
- $\mathcal{C} = \mathbf{C}\{a_1, \ldots, a_m\}$ and

$\mathcal{Q} = (0)$ so that V is smooth and equal to $\mathbf{C}^m$ or $(\mathbf{C}^m, 0)$, we have a generic Bernstein-Sato polynomial. It does not seem straigthforward to adapt her proof to the case where $\mathcal{Q} \neq (0)$ (i.e. when V is singular). Let us also say that she did not mention the fact that the polynomial she constructed is rational even though a detailed study of her proof shows that it is. As it appears, our main result covers and generalizes the previous existing results in this affine situation.

Notation. Let $\mathcal{P}$ be a prime ideal of $\mathcal{C}$. For c in $\mathcal{C}$, denote by $[c]_{\mathcal{P}}$ the class of c in the quotient $\mathcal{C}/\mathcal{P}$ and $(c)_{\mathcal{P}} = [c]_{\mathcal{P}}/1$ this class viewed in the fraction field of $\mathcal{C}/\mathcal{P}$. We naturally extend these notations to $\mathcal{C}[x]$, $\mathbf{A}_n(\mathcal{C})$ and $\mathcal{C}[x, 1/(f_1 \cdots f_p), s]$.

Remark. Using these notations, we can see that the polynomial $b(s)$ of Theorem 1 is a Bernstein-Sato polynomial of $(f)_{\mathcal{P}}$ for any $\mathcal{P} \in V(\mathcal{Q}) \setminus V(h)$. This justifies the name of a generic Bernstein-Sato polynomial on $V(\mathcal{Q})$.

As an application of Theorem 1, we obtain some consequences:

Corollary 2. *Fix a positive integer* d *and a field* $\mathbf{k}$. *For each* $j = 1, \ldots, p$, *take* $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$ *with* $\alpha \in \mathbf{N}^n$ *and* $a_{\alpha,j}$ *an indeterminate. Take* $a = (a_{\alpha,j})$ *for* $|\alpha| \leq d$ *and* $j = 1, \ldots, p$ *such that we see* $f = (f_1, \ldots, f_p)$ *in* $\mathbf{k}[a][x]^p$. *Denote by* m *the number of the* $a_{\alpha,j}$*'s.*

Then there exists a finite partition of $\mathbf{k}^m = \cup W$ *where each* W *is a locally closed subset of* $\mathbf{k}^m$ *(i.e.* W *is a difference of two Zariski closed sets) such that for any* W, *there exists a polynomial* $b_W(s) \in \mathbf{Q}[s_1, \ldots, s_p] \setminus 0$ *such that for each* a_0 *in* W, $b_W(s)$ *is in* $\mathcal{B}^v(f(a_0, x))$.

Remark.

- This corollary generalizes to the case $p \geq 2$ the main result of A. Leykin [Le] and J. Briançon and Ph. Maisonobe [Br-Ma] in the case $p = 1$.
- There is another way to generalize these results: Given a well ordering $<$ on $\mathbf{N}^p$ compatible with sums, it is possible to prove the existence of a partition $\mathbf{k}^m = \cup W$ into locally closed subsets with the following property: For any W, there exists a finite subset $G_W \subset \mathbf{k}[a][x]$ such that for any $a_0 \in W$, the set $G_W(a_0)$ is a $<$-Gröbner basis of the Bernstein-Sato ideal $\mathcal{B}^v(f(a_0, x))$, see [Br-Ma] and [Ba].

Proof of Corollary 2. We remark that we can give the same statement as in Corollary 2 for any algebraic subset $Y \subset \mathbf{k}^m$ as a space of parameters. The statement of Corollary 2 will then follow from the proof of this more general statement, that we shall give by an induction on the dimension of Y. If $\dim Y = 0$, the result is trivial. Suppose $\dim Y \geq 1$. Write $Y = V_m(\mathcal{Q}_1) \cup \cdots \cup V_m(\mathcal{Q}_r)$ where the $\mathcal{Q}_i$'s are prime ideals in $\mathbf{k}[a]$ (we identify the maximal ideals of $\mathbf{k}[a]$ and the points of $\mathbf{k}^m$). For each i, let $h_i \in \mathbf{k}[a] \setminus \mathcal{Q}_i$ and $b_i(s) \in \mathbf{Q}[s] \setminus 0$ be the h and $b(s)$ of Theorem 1 applied to $\mathcal{Q}_i$. Now, write

$$Y = \left(\bigcup_{i=1}^{r} V_m(\mathcal{Q}_i) \setminus V_m(h_i)\right) \bigcup Y',$$

with $Y' = \bigcup \big(V_m(\mathcal{Q}_i) \cap V_m(h_i)\big)$ for which $\dim Y' < \dim Y$. Apply the induction hypothesis to Y'. We obtain that Y is a union (not necessarily disjoint) of locally closed subsets V such that for each V there exists $b_V(s) \in \mathbf{Q}[s] \setminus 0$ which is in $\mathcal{B}^v(f(a_0, x))$ for any $a_0 \in V$. Let us show now how to obtain the annouced partition. Let B be the set of the obtained polynomials b_V's. Set $B = \{b_1, \dots, b_e\}$. For any $i = 1, \dots, e$, let E_i be the set of the V's for which $b_i = b_V$. Put

- $W_1 = \bigcup_{V \in E_1} V$,
- $W_2 = \Big(\bigcup_{V \in E_2} V\Big) \setminus \Big(\bigcup_{V \in E_1} V\Big)$,

  $\vdots$
- $W_e = \Big(\bigcup_{V \in E_e} V\Big) \setminus \Big(\bigcup_{V \in E_1 \cup \dots \cup E_{e-1}} V\Big)$.

Note that some of the W_i's may be empty. The set $\{(b_1, W_1), \dots, (b_e, W_e)\}$ gives a partition $Y = \cup W_i$ in a way that $b_i \in \mathcal{B}^v(f(a_0, x))$ for any $a_0 \in W_i$. $\square$

Corollary 3. *Take $f_1(a, x), \dots, f_p(a, x) \in \mathcal{O}(U)[x]$ where $\mathcal{O}(U)$ denotes the ring of holomorphic functions on a open subset U of $\mathbf{C}^m$.*

Then there exists a finite partition of $U = \cup W$ where each W is an (analytic) locally closed subset of U (i.e. each W is a difference of two analytic subsets of U) such that for any W, there exists a rational non zero polynomial $b(s)$ which belongs to $\mathcal{B}^v(f(a_0, x))$ for any $a_0 \in W$.

Remark. As it will appear in the proof, we have the same result if we replace $\mathcal{O}(U)$ by $\mathbf{C}\{a_1, \dots, a_m\}$ or $\mathbf{k}[[a_1, \dots, a_m]]$ ($\mathbf{k}$ being an arbitrary field).

Proof. Let us write $f_j(a, x) = \sum g_{\alpha,j}(a) x^\alpha$ where $g_{\alpha,j} \in \mathcal{O}(U)$. Let m be the number of the $g_{\alpha,j}$'s and let us introduce m new variables $b_{\alpha,j}$. Consider the (analytic) map $\phi : U \ni a \mapsto (b_{\alpha,j} = g_{\alpha,j}(a))_{\alpha,j} \in \mathbf{C}^m$. Now apply Corollary 2 to this situation. Let $\mathbf{C}^m = \cup W$ be the obtained partition and for any W, let $b_W \in \mathbf{Q}[s]$ be the polynomial given in 2. Now apply ϕ^{-1}. This gives a partition $U = \cup \phi^{-1}(W)$. Since ϕ is analytic, the sets $\phi^{-1}(W)$ are locally closed analytic subsets of U. It is then clear that for any W and $a_0 \in \phi^{-1}(W)$, we have $b_W \in \mathcal{B}^v(f(a_0, x))$. $\square$

Proof of the main result. In order to prove Theorem 1, we shall first prove the following.

Theorem 4. *Let $\mathbf{k}$ be a field and $f \in \mathbf{k}[x]^p$. Then $\mathcal{B}^v(f) \cap \mathbf{Q}[s]$ is not zero.*

Note that in [Br], the author proved (for $p = 1$) that the global Bernstein polynomial has rational roots for any field $\mathbf{k}$ of characteristic zero. The proof of 4 will use the following propositions.

Proposition 5. *Let $\mathbf{K}$ be a subfield of a field $\mathbf{L}$. Suppose that $f \in \mathbf{K}[x]^p$. Let $b(s) \in \mathbf{K}[s]$ be such that $b(s) f^s \in \mathbf{A}_n(\mathbf{L})[s] f^{s+v}$. Then*

$$b(s) f^s \in \mathbf{A}_n(\mathbf{K})[s] f^{s+v}.$$

Proof. The proof is inspired by [Br] in which the case $p = 1$ is treated. As $\mathbf{L}$ is a $\mathbf{K}$-vector space, let us take $\{1\} \cup \{l_\gamma;\, \gamma \in \Gamma\}$ as a basis so that $\mathbf{L}[x, s, 1/(f_1 \cdots f_p)] f^s$ is a free $\mathbf{K}[x, s, 1/(f_1 \cdots f_p)]$-module with $\{f^s\} \cup \{l_\gamma f^s;\, \gamma \in \Gamma\}$ as a basis. Now let P be in $\mathbf{A}_n(\mathbf{L})[s]$ such that $b(s) f^s = P f^{s+v}$. We decompose $P = P_0 + P'$ where $P_0 \in \mathbf{A}_n(\mathbf{K})[s]$ and P' has its coefficients in $\bigoplus_{\gamma \in \Gamma} \mathbf{K} \cdot l_\gamma$. Now, we have:

$$b(s) f^s = P_0 f^{s+v} + P' f^{s+v},$$

with $b(s) f^s$ and $P_0 f^{s+v}$ in $\mathbf{K}[x, s, 1/(f_1 \cdots f_p)] f^s$ and $P' f^{s+v}$ in $\bigoplus_{\gamma \in \Gamma} \mathbf{K}[x, s, 1/(f_1 \cdots f_p)] l_\gamma f^s$. By identification, we obtain:

$$b(s) f^s = P_0 f^{s+v}.$$

$\square$

Proposition 6 ([Br] and [Br-Ma2])**.** *Given $f \in \mathbf{C}[x]^p$, we have:*

(1) *The set $\{\mathcal{B}^v_{x_0}(f);\, x_0 \in \mathbf{C}^n\}$ is finite.*

(2) *$\mathcal{B}^v(f)$ is the intersection of all the $\mathcal{B}^v_{x_0}(f)$ where $x_0 \in \mathbf{C}^n$.*

Proof of Theorem 4. We shall divide the proof into two steps:

(a) First, suppose that $\mathbf{k} = \mathbf{C}$. By [Sa1, Sa2, Gy], as mentioned in the introduction, each $\mathcal{B}^v_{x_0}(f)$ contains a non zero rational polynomial. By the previous proposition, we can take a finite product of these polynomials and obtain a rational polynomial in $\mathcal{B}^v(f)$.

(b) Now suppose that $\mathbf{k}$ is arbitrary. Let $c_1, \dots, c_N$ be all the coefficients that appear in the writing of the f_j's and consider the field $\mathbf{K} = \mathbf{Q}(c_1, \dots, c_N)$. There exist $e_1, \dots, e_N \in \mathbf{C}$ and an injective morphism of fields $\phi : \mathbf{K} \to \mathbf{C}$ such that $\phi(c_i) = e_i$ for any i. We denote by the same symbol ϕ the natural extension of ϕ from $\mathbf{K}[x]$ to $\mathbf{C}[x]$ and from $\mathbf{A}_n(\mathbf{K})[s]$ to $\mathbf{A}_n(\mathbf{C})[s]$. Now, consider in $\mathbf{C}[s]$ the Bernstein-Sato ideal $\mathcal{B}^v(\phi(f))$ (where $\phi(f) = (\phi(f_1), \dots, \phi(f_p))$. Using the result of case (a), there exists $b(s) \in \mathbf{Q}[s] \smallsetminus 0$ that belongs to $\mathcal{B}^v(\phi(f))$. So we have a functional equation:

$$b(s)\phi(f)^s = P \cdot \phi(f)^{s+v},$$

where $P \in \mathbf{A}_n(\mathbf{C})[s]$. By Proposition 5, we can suppose $P \in \mathbf{A}_n(\phi(\mathbf{K}))[s]$. Apply ϕ^{-1} to this equation. Since $b(s) \in \mathbf{Q}[s]$, $\phi^{-1}(b(s)) = b(s)$, thus we obtain:

$$b(s)f^s = \phi^{-1}(P) \cdot f^{s+v}.$$

In conclusion $b(s)$ is in $\mathcal{B}^v(f)$. □

Now we dispose of a sufficient material to give the

Proof of Theorem 1. By Theorem 4, there exists a non zero rational polynomial $b(s)$ in $\mathcal{B}^v((f)_{\mathcal{Q}})$. Hence, we have the following equation:

$$b(s)\left(\frac{[f]_{\mathcal{Q}}}{1}\right)^s = \frac{[U(s)]_{\mathcal{Q}}}{[h]_{\mathcal{Q}}} \cdot \left(\frac{[f]_{\mathcal{Q}}}{1}\right)^{s+v},$$

where $U(s) \in \mathbf{A}_n(\mathcal{C})[s]$ and $h \in \mathcal{C} \smallsetminus \mathcal{Q}$. It follows that:

$$h\, b(s)f^s - U(s) \cdot f^{s+v} \equiv 0 \bmod \mathcal{Q}$$

in $\mathcal{C}[x, 1/(f_1, \dots, f_p), s]f^s$. Since $f_1 \cdots f_p \notin \mathcal{Q}[x]$ and $\mathcal{Q}$ is prime, we obtain:

$$h\, b(s)f^s - U(s) \cdot f^{s+v} \in \mathcal{Q}\Big[x, \frac{1}{f_1 \cdots f_p}, s\Big]f^s.$$

□

This article is a more general and simplified version of some results of my thesis [Ba].

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