A note on the mean value of the zeta and L-functions. XIII

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Abstract: Extending the discussion in the previous note [6] of this series, the group $\mathrm{PSL}_2(\mathbf{C})$ will be dealt with in place of $\mathrm{PSL}_2(\mathbf{R})$. We shall indicate that the functional structure that supports the spectral theory of Kloosterman sums in the complex case is essentially the same as in the real case, though it is more involved as can be expected.

Key words: Mean values of zeta-functions; local functional equations of Jacquet-Langlands; Gamma-functions of representations; Bessel functions of representations.

1. Introduction. The aim of the present note is to show that the argument in [6] extends to the situation $G = \mathrm{PSL}_2(\mathbf{C})$. Thus, we are going to prove the complex analogues of Theorems 1 and 2 of [6]. In themselves, these theorems are results in the analysis on the Lie group G. Our interest stems from their use in the spectral theory of Kloosterman sums, as developed in [1]. This application works for all discrete subgroups of G. It should, however, be noted that the resulting spectral decomposition of the fourth power moment of Dedekind zeta-functions — carried out in [1] — has been obtained so far only in the case where the underlying imaginary quadratic number field is of class number one (see Section 14 of [1], Note X [2], and the concluding remark below). We shall use basic facts on G as a Lie group, which can be found in [1] with greater details. Otherwise this paper is essentially self-contained.

2. Basic concepts. We have first to define a coordinate system on G: Put

$$\mathrm{n}[z] = \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \quad \mathrm{h}[u] = \begin{bmatrix} u & \\ & 1/u \end{bmatrix}, \quad \mathrm{k} = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

for $z, u, \alpha, \beta \in \mathbf{C}$, $u \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, where the matrices are understood projectively. Also put

$$N = \{\mathrm{n}[z] : z \in \mathbf{C}\}, \ A = \{\mathrm{a}[r] : r > 0\},$$
$$K = \mathrm{PSU}(2) = \{\mathrm{k}[\alpha, \beta] : \alpha, \beta \in \mathbf{C}\},$$

where $\mathrm{a}[r] = \mathrm{h}[\sqrt{r}]$. With the Euler angles φ, θ, ψ, we have

$$\mathrm{k}[\alpha, \beta] = \mathrm{h}[e^{i\varphi/2}]\mathrm{k}[\cos(\theta/2), i\sin(\theta/2)]\mathrm{h}[e^{i\psi/2}].$$

The Iwasawa decomposition of G is

$$G = NAK,$$

which is read as $\mathrm{g} = \mathrm{nak} = \mathrm{n}[z]\mathrm{a}[r]\mathrm{k}[\alpha, \beta]$. The Haar measures on respective groups are normalized by

$$d\mathrm{n} = d\,\mathrm{Re}\, z\, d\,\mathrm{Re}\, z, \quad d\mathrm{a} = r^{-1}dr,$$
$$d\mathrm{k} = (8\pi^2)^{-1}\sin\theta d\varphi d\theta d\psi$$

and

$$d\mathrm{g} = r^{-2}d\mathrm{n}d\mathrm{a}d\mathrm{k}.$$

Let $L^2(\Gamma\backslash G)$ be the Hilbert space composed of all left Γ-automorphic functions on G which are square integrable with respect to the measure $d\mathrm{g}$. We have

$$L^2(\Gamma\backslash G) = \mathbf{C} \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^eL^2(\Gamma\backslash G).$$

The cuspidal subspace ${}^0L^2(\Gamma\backslash G)$ is spanned by all functions in $L^2(\Gamma\backslash G)$ with vanishing constant terms in their Fourier expansions relative to the left action of N; and ${}^eL^2(\Gamma\backslash G)$ is generated by integrals of Eisenstein series. We have the decomposition into right-irreducible subspaces:

$${}^0L^2(\Gamma\backslash G) = \overline{\bigoplus V}.$$

To classify representations V, we need two Casimir elements Ω_+, $\Omega_- = \overline{\Omega_+}$, that generate the center of the universal enveloping algebra of G. We have

$$\Omega_+ = \frac{1}{2}r^2\partial_z\partial_{\overline{z}} + \frac{1}{2}re^{i\varphi}\cot\theta\partial_z\partial_\varphi - \frac{1}{2}ire^{i\varphi}\partial_z\partial_\theta$$

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$$- \frac{re^{i\varphi}}{2\sin\theta}\partial_z\partial_\psi + \frac{1}{8}r^2\partial_r^2 - \frac{1}{4}ir\partial_r\partial_\varphi$$
$$- \frac{1}{8}\partial_\varphi^2 - \frac{1}{8}r\partial_r + \frac{1}{4}i\partial_\varphi.$$

They become constant multiplications in each V: There are $p \in \mathbf{Z}$, $\kappa \in \mathbf{R}$ such that

$$\Omega_\pm|_V = -\chi_V^\pm \cdot 1, \quad \chi_V^\pm = \frac{1}{8}((\kappa \pm ip)^2 + 1).$$

We call the pair (p, κ) the spectral parameter of V.

According to the right action of K, the space V decomposes into K-irreducible subspaces

$$V = \overline{\bigoplus_{|p|\le l} V_l}, \quad \dim V_l = 2l + 1.$$

To describe this precisely, let Ω_K be a Casimir element of the universal enveloping algebra of K:

$$\Omega_K = \frac{1}{2\sin^2\theta}\Big(\partial_\varphi^2 + \sin^2\theta\partial_\theta^2 + \partial_\psi^2 - 2\cos\theta\partial_\varphi\partial_\psi + \sin\theta\cos\theta\,\partial_\theta\Big).$$

Then we have

$$V_l = \bigoplus_{q=-l}^{l} V_{l,q}, \quad \dim V_{l,q} = 1,$$

with

$$V_{l,q} = \left\{ f \in V : \Omega_K f = -\frac{1}{2}l(l+1),\ \partial_\psi f = -iqf \right\}.$$

Any non-zero element of $V_{l,q}$ is called a left Γ-automorphic form of spectral parameter (p, κ) and K-type (l, q).

To have a convenient way to exhibit the representation V, we introduce functions $\Phi_{p,q}^l$ on K, which are defined via

$$F_q(x, \mathrm{k}[\alpha, \beta]) = (\alpha x - \overline{\beta})^{l-q}(\beta x + \overline{\alpha})^{l+q} = \sum_{p=-l}^{l} \Phi_{p,q}^l(\mathrm{k})x^{l-p}.$$

The system $\{\Phi_{p,q}^l : |p|, |q| \le l\}$ is a complete orthogonal basis of $L^2(K)$ with norms

$$\|\Phi_{p,q}^l\|_K = \frac{1}{\sqrt{l+1/2}}\left(\frac{\Gamma(l-q+1)\Gamma(l+q+1)}{\Gamma(l-p+1)\Gamma(l+p+1)}\right)^{1/2}.$$

We use the following functions on G:

$$\phi_{l,q}(\mathrm{nak}) = \frac{\Phi_{p,q}^l(\mathrm{k})}{\|\Phi_{p,q}^l\|_K} r^{1+i\kappa}.$$

Let

$$\varphi(\mathrm{g}) = \sum_{\omega\ne 0} W_\omega(\mathrm{g}; \varphi), \quad \omega \in \mathbf{Z}[i]$$

be the Fourier expansion of $\varphi \in V$ with respect to the left action of N. If $\varphi \in V_{l,q}$, then $W_\omega(\mathrm{g}; \varphi)$ is a multiple of the Jacquet transform

$$\mathcal{A}_\omega\phi_{l,q}(\mathrm{g}) = \int_{\mathbf{C}} \exp(-2\pi i \operatorname{Re}(\omega\lambda))\phi_{l,q}(\mathrm{wn}[z]\mathrm{g})d\operatorname{Re}\lambda\, d\operatorname{Im}\lambda,$$

with

$$\mathrm{w} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

One can choose generating vectors $\varphi_{l,q}$ of the spaces $V_{l,q}$ that transforms under the action of the Lie algebra in the same way as the $\phi_{l,q}$, so that $\{\varphi_{l,q} : |p| \le l, |q| \le l\}$ is a complete orthonormal system in V. Note that p is fixed by V, and l, q run over integers as indicated. With this, we may put

$$W_\omega(\mathrm{g}; \varphi_{l,q}) = |\omega|^{-i\kappa}\varrho_V(\omega)\Gamma(1 + i\kappa)\mathcal{A}_\omega\phi_{l,q}(\mathrm{g}).$$

Then $\varrho_V(\omega)$ depends only on V, ω, and is called the Fourier coefficient of the representation V.

3. Unitaricity. We are now ready to prove the complex analogue of Theorem 1 of [6]. This confirms the unitary nature of the Kirillov model for unitary principal series representations of G.

Theorem 1. *Let φ be an arbitrary smooth vector in V, and $W_\omega(\mathrm{g}; \varphi)$ its Fourier coefficient with respect to the left action of N. Then we have*

$$\int_{\mathbf{C}^\times} |W_\omega(\mathrm{h}[u]; \varphi)|^2 d^\times u = \left(\frac{\pi}{2|\omega|}\right)^2 \frac{\kappa}{\sinh \pi\kappa}|\varrho_V(\omega)|^2\|\varphi\|_{\Gamma\backslash G}^2,$$

where $\mathbf{C}^\times = \mathbf{C}\backslash\{0\}$ and $d^\times u = |u|^{-2}d\operatorname{Re}u\, d\operatorname{Im}u$.

Proof. It is sufficient to show the orthogonality relation:

$$\int_{\mathbf{C}^\times} W_\omega(\mathrm{h}[u]; \varphi_{l,q})\overline{W_\omega(\mathrm{h}[u]; \varphi_{l',q'})}d^\times u = \delta_{l,l'}\delta_{q,q'}\left(\frac{\pi}{2|\omega|}\right)^2 \frac{\kappa}{\sinh \pi\kappa}|\varrho_V(\omega)|^2.$$

To this end we note that from the above definition follows

$$\mathcal{A}_\omega\phi_{l,q}(\mathrm{g}) = \frac{\exp(2\pi i \operatorname{Re}\omega z)}{\|\Phi_{p,q}^l\|_K} \sum_{|m|\le l} v_m^l(r, \omega)\Phi_{m,q}^l(\mathrm{k}),$$

where

$$(1) \qquad v_m^l(r,\omega) = r^{1-i\kappa}\int_{\mathbf{C}} \frac{\exp(-2\pi i \operatorname{Re}\omega r\lambda)}{(1+|z|^2)^{1+i\kappa}} \cdot \Phi_{p,m}^l\left(\mathrm{k}\left[\frac{\overline{\lambda}}{\sqrt{1+|\lambda|^2}}, \frac{-1}{\sqrt{1+|\lambda|^2}}\right]\right) d\operatorname{Re}\lambda\, d\operatorname{Im}\lambda$$
$$= 2\pi r^{1-i\kappa}\left(\frac{i\omega}{|\omega|}\right)^{-p-m}\int_0^\infty \xi\frac{J_{p+m}(2\pi|\omega|r\xi)}{(1+\xi^2)^{1+i\kappa}} \cdot \Phi_{p,m}^l\left(\mathrm{k}\left[\frac{\xi}{\sqrt{1+\xi^2}}, \frac{-1}{\sqrt{1+\xi^2}}\right]\right) d\xi,$$

with J_p being the J-Bessel function of order p (see Section 5 of [1] for details). The first expression gives

$$\mathcal{A}_\omega\phi_{l,q}(\mathrm{h}[u]) = \|\Phi_{p,q}^l\|_K^{-1} \cdot |\omega|^{i\kappa-1}(\omega/|\omega|)^{-p-q}v_q^l(|\omega||u|^2,1)(u/|u|)^{-2q}.$$

Thus

$$\int_{\mathbf{C}^\times} W_\omega(\mathrm{h}[u];\varphi_{l,q})\overline{W_\omega(\mathrm{h}[u];\varphi_{l',q'})}d^\times u = \pi\delta_{q,q'}\|\Phi_{p,q}^l\|_K^{-1}\|\Phi_{p,q'}^{l'}\|_K^{-1} \cdot |\Gamma(1+i\kappa)|^2|\omega|^{-2}|\varrho_V(\omega)|^2\int_0^\infty v_q^l(r)\overline{v_{q'}^{l'}(r)}\frac{dr}{r},$$

where $v_m^l(r) = v_m^l(r,1)$.

To compute the last integral, we argue as (9) of [6]: The functions $v_q^l(r)$ satisfy the differential equations

$$D_q^+v_q^l(r) = -4\pi i(l-q)r^{-1}v_{q+1}^l(r),$$
$$D_q^-v_q^l(r) = 4\pi i(l+q)r^{-1}v_{q-1}^l(r),$$

where $D_q^- = \overline{D_{-q}^+}$ with

$$D_q^+ = \left(\frac{d}{dr}\right)^2 - (2q+1)r^{-1}\frac{d}{dr} + r^{-2}(q^2+2q-4\pi^2r^2-8\chi_V^+).$$

This is equivalent to $\Omega_\pm\varphi_{l,q} = -\chi_V^\pm\varphi_{l,q}$ (see either (4.14)–(4.15) of [1] or p. 236 of [4]). Thus the integral is equal to

$$-\frac{1}{4\pi i(l-q+1)}\int_0^\infty D_{q-1}^+v_q^l(r)\cdot\overline{v_q^{l'}(r)}dr = -\frac{1}{4\pi i(l-q+1)}\int_0^\infty v_{q-1}(r)\cdot\overline{D_q^-v_q^{l'}(r)}dr = \frac{l+q}{l-q+1}\int_0^\infty v_{q-1}^l(r)\overline{v_{q-1}^{l'}(r)}\frac{dr}{r}.$$

This procedure is valid only if $v_q^l(r)$ tends to 0 sufficiently fast as r tends to either 0 or ∞, which in fact follows from the first expression in (1). Hence

$$\int_0^\infty v_q^l(r)\overline{v_q^{l'}(r)}\frac{dr}{r} = \frac{l-q}{l+q+1}\int_0^\infty v_{q+1}^l(r)\overline{v_{q+1}^{l'}(r)}\frac{dr}{r} = \delta_{l,l'}\binom{2l}{l-q}^{-1}\int_0^\infty |v_l^l(r)|^2\frac{dr}{r}.$$

Via Lemma 5.1 of [1], or directly from the second expression in (1), we have

$$|v_l^l(r)| = 2\frac{(\pi r)^{l+1}}{|\Gamma(l+1+i\kappa)|}\binom{2l}{l-p}|K_{p+i\kappa}(2\pi r)|;$$

thus

$$\int_0^\infty |v_l^l(r)|^2\frac{dr}{r} = \frac{1}{4(l+\frac{1}{2})}\binom{2l}{l-p}.$$

This ends the proof of Theorem 1. □

4. Bessel transform. We are turning to the complex analogue of Theorem 2 of [6]. We shall carry out the line set in [6]; that is, a combination of the Mellin transform and the local functional equation for the complex case will be employed.

Theorem 2. *Let V, φ, and W_ω be as in the previous theorem. Then we have*

$$W_1(\mathrm{h}[u];\varphi(*\cdot\mathrm{w})) = \int_{\mathbf{C}^\times} j_V(uv)W_1(\mathrm{h}[v];\varphi)d^\times v,$$

with

$$(2) \qquad j_V(u) = -2\pi^2\frac{|u|^2}{\sin\pi i\kappa} \cdot [\mathcal{J}_{i\kappa,p}(2\pi u) - \mathcal{J}_{-i\kappa,-p}(2\pi u)].$$

Here (p,κ) is the spectral parameter of V, and

$$\mathcal{J}_{\nu,p}(u) = |u/2|^{2\nu}(u/|u|)^{-2p}J_{\nu-p}^*(u)J_{\nu+p}^*(\overline{u}),$$

with $J_\nu^*(u)$ being the branch of $J_\nu(u)(u/2)^{-\nu}$ such that $J_\nu^*(0)=1$.

Remark. For arbitrary non-zero $\omega\in\mathbf{Z}[i]$, the integral representation for $W_\omega(h[u];\varphi(*\cdot\mathrm{w}))$ follows from the relation

$$\varrho_V(1)W_\omega(\mathrm{h}[u];\varphi) = \frac{\varrho_V(\omega)}{\omega}\left(\frac{\omega}{|\omega|}\right)^{-p}W_1(\mathrm{h}[\sqrt{\omega}u];\varphi).$$

Also note that the function j_V depends only on the spectral parameter; thus only on the isomorphism class of the representation V.

Proof. It is enough to treat only the weight vectors $\varphi=\varphi_{l,q}$. According to Theorem 12.1 of [1] (see also Theorem 3 of [2]), we have, for $u=|u|e^{i\vartheta}$, $\vartheta\in\mathbf{R}$,

$$(3)\quad j_V(u) = (-1)^p 4\pi|u|^2 \cdot \int_0^\infty y^{2i\kappa-1}\left(\frac{ye^{i\vartheta}+(ye^{i\vartheta})^{-1}}{|ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|}\right)^{2p} \cdot J_{2p}(2\pi|u||ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|)dy.$$

Note that the equality between the right sides of these two expressions for j_V holds for $|\operatorname{Im}\kappa| < \frac{1}{4}$, via analytic continuation. We shall prove the assertion of the theorem with this expression for j_V. Later we shall give a simple proof of (3).

Thus, let us consider the integral

$$\Gamma_{l,q}(s) = \int_0^\infty v_q^l(r) r^{2(s-1)} dr.$$

The second expression in (1) gives

$$\Gamma_{l,q}(s) = i^{-p-q}\pi^{1+i\kappa-2s}(-1)^{\min(0,p+q)} \cdot \frac{\Gamma(s+\frac{1}{2}(|p+q|-i\kappa))}{\Gamma(1-s+\frac{1}{2}(|p+q|+i\kappa))}\mathrm{L}_{l,q}(s),$$

with

$$\mathrm{L}_{l,q}(s) = \int_0^\infty \frac{\xi^{1+i\kappa-2s}}{(1+\xi^2)^{1+i\kappa}} \cdot \Phi_{p,q}^l\left(\mathrm{k}\left[\frac{\xi}{\sqrt{1+\xi^2}}, \frac{-1}{\sqrt{1+\xi^2}}\right]\right)d\xi.$$

This is a counterpart of L_p of [6]. We have, analogously,

$$\mathrm{L}_{l,q}(s) = (-1)^{l+p}\mathrm{L}_{l,-q}(1-s),$$

which is again a result of the change of variable $\xi \to \xi^{-1}$. Note that we have used the fact $\Phi_{-p,-q}^l = (-1)^{p+q}\overline{\Phi_{p,q}^l}$, which is equivalent to

$$\overline{F_q(x,\mathrm{k})} = (-1)^{l-q}x^{2l}F_{-q}(-1/x,\mathrm{k}),$$

with F_q as above. The absolute convergence necessary in these derivations and the meromorphic continuation to $\mathbf{C}$ of $\mathrm{L}_{l,q}(s)$ can be confirmed as before. Hence we have the local functional equation

$$(-1)^{l-q}\Gamma_{l,-q}(s) = \pi^{2-4s}(-1)^{\max(|p|,|q|)}\Gamma_{l,q}(1-s) \cdot \frac{\Gamma(s+\frac{1}{2}(|p+q|+i\kappa))\Gamma(s+\frac{1}{2}(|p-q|-i\kappa))}{\Gamma(1-s+\frac{1}{2}(|p+q|-i\kappa))\Gamma(1-s+\frac{1}{2}(|p-q|+i\kappa))}$$

(see Theorem 6.4 of [4]). With this, observe that

$$\int_0^\infty y^{2\nu-1}J_{|p+q|}(ry)J_{|p-q|}(r/y)dy \longleftrightarrow \frac{2^{s-3}\Gamma(\frac{1}{4}s+\frac{1}{2}(|p+q|+\nu))\Gamma(\frac{1}{4}s+\frac{1}{2}(|p-q|-\nu))}{\Gamma(1-\frac{1}{4}s+\frac{1}{2}(|p+q|-\nu))\Gamma(1-\frac{1}{4}s+\frac{1}{2}(|p-q|+\nu))}$$

is a Mellin pair, provided $2|\operatorname{Re}\nu| < \operatorname{Re}s < 1 - 2|\operatorname{Re}\nu|$. Thus, denoting the left side by $K_{\nu,p}(r,q)$, we get, by the Mellin-Parseval formula,

$$(-1)^{l-q}y^{-2}v_{-q}^l(y^2) = 8\pi^2(-1)^{\max(|p|,|q|)} \cdot \int_0^\infty K_{i\kappa,p}(2\pi yr,q)v_q^l(r^2)rdr.$$

This is equivalent to

$$W_1(\mathrm{h}[u];\varphi_{l,q}(*\cdot\mathrm{w})) = 4\pi|u|^2\int_{\mathbf{C}^\times}|v|^2W_1(\mathrm{h}[v];\varphi_{l,q}) \cdot \sum_{\mu\in\mathbf{Z}}(-1)^{\max(|p|,|\mu|)}K_{i\kappa,p}(2\pi|uv|,\mu)\left(\frac{uv}{|uv|}\right)^{2\mu}d^\times v.$$

We apply Graf's addition theorem (the formula (1) on p. 359 of [7]) to the last sum, and conclude the proof of the theorem. As for the treatment of convergence issues involved in the last steps, see Lemma 4 of [5] and the discussion prior to Lemma 14.1 of [1].

We move to a proof of (3). The proof given in [1] of the equivalence between (2) and (3) depends on the concept of the Goodman-Wallach operator [3]. We shall supply here a proof which is conceptually much simpler: The procedure treating the last sum over μ means in effect that the $2q$-th Fourier coefficient of the right side of (3) is equal to $4\pi(-1)^{\max(|p|,|q|)}|u|^2K_{i\kappa,p}(2\pi|u|,q)$. We are going to show that the $2q$-th Fourier coefficient of the right side of (2) is equal to this quantity, which should end the proof.

Thus, the last Mellin pair implies that we have

$$K_{\nu,p}(r,q) = \frac{1}{16\pi i}\int_{(\eta)}\frac{\Gamma(\frac{1}{4}s+\frac{1}{2}(\nu+|p+q|))}{\Gamma(1-\frac{1}{4}s+\frac{1}{2}(|p+q|-\nu))} \cdot \frac{\Gamma(\frac{1}{4}s-\frac{1}{2}(\nu-|p-q|))}{\Gamma(1-\frac{1}{4}s+\frac{1}{2}(\nu+|p-q|))}\left(\frac{r}{2}\right)^{-s}ds,$$

with $2|\operatorname{Re}\nu| < \eta < 1$. Take the contour to $-\infty$. Residue calculus gives that

$$(-1)^{\max(|p|,|q|)}\frac{2}{\pi}K_{\nu,p}(r,q) = -\frac{(-1)^{p+q}}{\sin\pi\nu}\sum_{k=0}^\infty\frac{(r/2)^{2\nu+2|p+q|+4k}}{\Gamma(k+1)\Gamma(k+1+|p+q|)} \cdot \frac{1}{\Gamma(\nu+k+1+\frac{1}{2}(|p+q|+|p-q|))} \cdot \frac{1}{\Gamma(\nu+k+1+\frac{1}{2}(|p+q|-|p-q|))} + \frac{(-1)^{p+q}}{\sin\pi\nu}\sum_{k=0}^\infty\frac{(r/2)^{-2\nu+2|p-q|+4k}}{\Gamma(k+1)\Gamma(k+1+|p-q|)}$$

$$\cdot \frac{1}{\Gamma(-\nu+k+1+\frac{1}{2}(|p+q|+|p-q|))}$$
$$\cdot \frac{1}{\Gamma(-\nu+k+1-\frac{1}{2}(|p+q|-|p-q|))}.$$

This right side coincides with the coefficient of $(u/|u|)^{2q}$ ($|u|=r$) that can be read from the power series expansion of $-(\mathcal{J}_{\nu,p}(u)-\mathcal{J}_{-\nu,-p}(u))/\sin\pi\nu$, which proves our claim. □

Concluding remark. This corresponds to the ending part of [6], and concerns the spectral decomposition of the fourth power moment of the Dedekind zeta-function ζ_F of the Gaussian number field:

$$\int_{-\infty}^{\infty} |\zeta_F(1/2+it)|^4 g(t)dt,$$

where g is supposed, for instance, to be entire and of rapid decay in any fixed horizontal strip. According to either Theorem 14.1 of [1] or Theorem 5 of [2], the cuspidal part of the decomposition has the form

$$\sum_V \alpha_V H_V(1/2)^3 \Theta_V(g),$$

where V is as above, and assumed to be Hecke invariant; the H_V is the Hecke series associated with V, and α_V a metrical normalization factor. Our interest is in the construction of the functional Θ_V. From the formulas (2.32), (12.26), and (14.1) of [1], we see, after some elementary manipulation, that

$$\Theta_V(g) = \int_{\mathbf{C}} \frac{\hat{g}(2\log|1+1/u|)}{|u(u+1)|} \cdot \Xi_V(u)d\,\mathrm{Re}\,u\,d\,\mathrm{Im}\,u \tag{4}$$

with

$$\Xi_V(u) = \frac{1}{16\pi}\int_{\mathbf{C}^\times} j_V\left(\sqrt{v/u}\right) j_E\left(\sqrt{-v}\right)\frac{d^\times v}{|v|}. \tag{5}$$

Here

$$\hat{g}(x) = \int_{-\infty}^{\infty} g(t)e^{ixt}dt,$$

and, for $u=|u|e^{i\vartheta}$, $\vartheta\in\mathbf{R}$,

$$j_E(u) = 4\pi|u|^2\int_0^\infty J_0(2\pi|u||ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|)\frac{dy}{y},$$

which is the case $(p,\kappa)=(0,0)$ in (3). The formulas (4) and (5) correspond, respectively, to (15) and (13) of [6]. The similarity between them is indeed remarkable.

Corrigendum to Note XII. In the second equation on p. 39 of [6], the right side should be multiplied by the factor $|m|^{-1/2}$. Also in the adjacent equations the factors $\varrho_V(n)$ and $\varrho_V(m)$ are to be replaced by $|n|^{-1/2}\varrho_V(|n|)$ and $|m|^{-1/2}\varrho_V(|m|)$, respectively.

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