A note on the mean value of the zeta and L-functions. XII

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Abstract: In the present and the next notes of this series, we shall try to illuminate a geometric structure behind the interactions that have recently been observed between mean values of zeta-functions and automorphic representations. Our discussion is hoped to be a precursor of a unified theory of mean values of automorphic L-functions that we are going to forge. In this note we shall deal with the spectral structure over the modular group. In the next note the Picard group will be treated, as a typical case in the complex situation. We stress that we have been inspired by the work [2] due to Cogdell and Pyatetskii-Shapiro.

Key words: Mean values of zeta-functions; local functional equations of Jacquet–Langlands; Gamma functions of representations; Bessel functions of representations.

1. Introduction. The functional equation for the Riemann zeta-function $\zeta(s)$ and the Poisson summation formula over $\mathbf{Z}$ are equivalent. The theory of unitary representations puts the latter as a means to compute, in terms of seed functions, projections of Poincaré series over $\mathbf{Z}\backslash\mathbf{R}$ to irreducible subspaces of $L^2(\mathbf{Z}\backslash\mathbf{R})$. The kernel function, i.e., $\cos(2\pi x)$, $x \in \mathbf{R}$, of the relevant integral transform of seed functions is precisely the Mellin inverse of the Gamma-factor of the functional equation for $\zeta(s)$. Generalizing the setting, given a pair of a Lie group G and its discrete subgroup Γ, one may wonder what the analogue will be. The central problem is, of course, to find a way to compute explicitly the projection of a given Poincaré series over $\Gamma\backslash G$ to an arbitrary irreducible subspace of $L^2(\Gamma\backslash G)$. The above suggests that the solution should be closely related to the Gamma and Bessel functions of representations of G. Here we shall report about the case $G = \mathrm{PSL}_2(\mathbf{R})$ and $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, which is the most basic in applications in Analytic Number Theory. It will be observed that in place of a single functional equation for the pair $\mathbf{R}$ and $\mathbf{Z}$ a family of infinitely many functional equations, interlaced with the spectral resolution of the Casimir operator, govern the whole structure. This seems to lead us to a better understanding of the explicit spectral decomposition for the fourth moment of $\zeta(s)$, which is established in [6]. We shall, however, leave the report on this aspect of our investigation for future notes, and here rather concentrate on the description of basic principles except brief observations given in Concluding remark below.

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Our discussion is related to that of [2], but can be read independently. Ours is closer than theirs to the traditional treatment of problems in Analytic Number Theory concerning functional equations of L-functions, which are perhaps best represented by Voronoï formulas. In fact, the geometric structure we are describing is an extension of the Voronoï scheme to $\mathrm{PSL}_2(\mathbf{R})$. Details of our discussion below are available in [7], which is intended for publication.

2. Motivation. The author has been trying to have a direct proof of his result on the fourth power moment of $\zeta(s)$

$$\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 g(t)dt, \tag{1}$$

where g is supposed to be sufficiently smooth. Here the word direct suggests that it is wished to find a way to connect this moment with the space $L^2(\Gamma\backslash G)$ without appealing to the spectral theory of Kloosterman sums that is used in [6]. Since all the spectral data of irreducible subspaces of $L^2(\Gamma\backslash G)$ are present in the spectral decomposition of (1), while no trace of the use of Kloosterman sums is remaining there, it is reasonable to envisage the existence of such a way. An aim of the present article is to indicate an approach to this problem; as remarked above, actual discussions are, however, to be developed in our later works. It will amount to a realization of the programme introduced in Section 4.2 of [6].

The programme is, in hindsight, equivalent to finding as directly as possible the projection, to each irreducible subspace of $L^2(\Gamma\backslash G)$, of a Poincaré series of the following type:

$$P_f(\mathrm{g}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma\mathrm{g}),$$

where $\Gamma_\infty = \{\mathrm{n}[x] : x \in \mathbf{Z}\}$, and the smooth f defined over G is to satisfy $f(\mathrm{n}[x]\mathrm{g}) = e(mx)f(\mathrm{g})$, with $m \in \mathbf{Z}$, $e(x) = \exp(2\pi ix)$, and

$$\mathrm{n}[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}.$$

Note that the convergence issue is ignored in this section. Let V be a cuspidal irreducible subspace of $L^2(\Gamma\backslash G)$, and $\varpi_V(f)$ be the projection of P_f to V. We try to see $\varpi_V(f)$ via its Fourier expansion:

$$\varpi_V(f)(\mathrm{g}) = \sum_{n\in\mathbf{Z}} W_n(\mathrm{g}; \varpi_V(f)),$$

where

$$W_n(\mathrm{g}; \varphi) = \int_0^1 e(-nx)\varphi(\mathrm{n}[x]\mathrm{g})dx.$$

To compute $W_n(\mathrm{g}; \varpi_V(f))$ we proceed weakly, following Chapter 5 of [2], with our own subsequent twists. Thus, we pick up an arbitrary smooth vector φ of V and consider

$$\text{(2)} \qquad \int_{\mathbf{R}^\times} w_n(u; \varpi_V(f))\overline{w_n(u; \varphi)}d^\times u,$$

where

$$w_n(u; \varphi) = W_{n\ \mathrm{sgn}(u)}(\mathrm{a}[|u|]; \varphi), \quad \mathbf{R}^\times = \mathbf{R}\backslash\{0\},$$

$$d^\times u = \frac{du}{|u|}; \quad \mathrm{a}[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix}.$$

This is to use the Kirillov model realizing V in the space $L^2(\mathbf{R}^\times, d^\times)$ via the map $w_n(u; \varphi) \mapsto w_n(u; \varphi(* \cdot \mathrm{g}))$, if we adopt the terminology in [2]. A feature of this model, particularly important for our purpose, is in the unitaricity relation stated in Theorem 1 below. We shall prove the assertion by reducing it to a simple orthogonality relation among Whittaker functions, which in effect lets us dispense with the notion of the Kirillov model. At any event, the unitaricity relation transforms (2) into

$$c_V(n)\int_{N\backslash G} f(\mathrm{g})\overline{W_m(\mathrm{g}; \varphi)}d\mathrm{g},$$

where $c_V(n)$ is a constant, and $N = \{\mathrm{n}[x] : x \in \mathbf{R}\}$. This shows that our problem has been reduced to finding a kernel function $j_V^{m,n}(u; \mathrm{g})$ such that for any smooth vector φ in V

$$\text{(3)} \qquad W_m(\mathrm{g}; \varphi) = \int_{\mathbf{R}^\times} j_V^{m,n}(u; \mathrm{g}) w_n(u; \varphi)d^\times u.$$

Such a kernel has been known as the Bessel function of the representation V, and belongs to the family of reproducing kernels. Arguing formally, we obtain

$$w_n(u; \varpi_V(f)) = c_V(n)\int_{N\backslash G} f(\mathrm{g})\overline{j_V^{m,n}(u; \mathrm{g})}\, d\mathrm{g},$$

which should settle our current issue on P_f.

However, the equation (3) cannot hold in general with an ordinary function $j_V^{m,n}$. If g is in the small Bruhat cell, then $j_V^{m,n}$ is to be understood in terms of the Dirac delta, as is to be explained later. On the other hand, if g is in the big cell, then (3) can readily be reduced to the case $m = n = 1$ and $\mathrm{g} = \mathrm{w}$ the Weyl element, again to be made precise later. Remarkably, the kernel $j_V^{1,1}(u; \mathrm{w})$ has been given, in the form of an ordinary function, already in the formula (17) on p. 454 of [9] (see also the end of Chapter VII of [8] as well as Theorem 4.1 and Chapter 6 of [2]). It coincides exactly with that in the famed Kloosterman–Spectra sum formula due to N. V. Kuznetsov, a fact which illuminates the structure supporting his Bessel transform.

Although their claims on $j_V^{1,1}(u; \mathrm{w})$ are correct, the relevant discussions in [2, 8, 9] are by no means rigorous. Therefore, we shall develop, in Section 4, an independent argument to fix the kernel $j_V^{1,1}(u; \mathrm{w})$. For this sake, we shall appeal to a principle, which deals with self-reciprocal kernels. Thus, we shall apply the Mellin transform to translate (3) into a functional equation. There arises a necessity to invoke the local functional equation attached to the representation V, in the sense of [4]. We shall prove the latter in an elementary manner, for the sake of completeness, and finish the discussion with an appeal to the Mellin–Parseval formula.

Hence our purpose is not to claim any new results but to set forth a rigorous argument to support important claims made previously by others and acquire a new viewpoint for our own result on (1).

3. Basic concepts. We shall collect notions which are needed in the next section. Thus, let $G =$

NAK be the Iwasawa decomposition, where N is as above, and $A = \{\mathrm{a}[y] : y > 0\}$, $K = \{\mathrm{k}[\theta] : \theta \in \mathbf{R}/(\pi\mathbf{Z})\}$, with

$$\mathrm{k}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The Haar measure we use is $d\mathrm{g} = d\mathrm{n}d\mathrm{a}d\mathrm{k}/y$ with $\mathrm{g} = \mathrm{nak}$, $d\mathrm{n} = dx$, $d\mathrm{a} = dy/y$, $d\mathrm{k} = d\theta/\pi$. On this coordinate system the Casimir operator over G takes the form $\Omega = -y^2((\partial/\partial x)^2 + (\partial/\partial y)^2) + iy\partial^2/(\partial x \partial\theta)$.

Elements of G act on the functions in $L^2(\Gamma\backslash G)$ from the right, and we have the orthogonal decomposition to invariant subspaces: $L^2(\Gamma\backslash G) = \mathbf{C}\cdot 1 \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^eL^2(\Gamma\backslash G)$. The ${}^eL^2$ is spanned by integrals of Eisenstein series. The ${}^0L^2$ is the cuspidal subspace which is spanned by square-integrable left Γ-automorphic functions whose Fourier expansions in x have constant terms vanishing. It decomposes into right-irreducible subspaces: ${}^0L^2(\Gamma\backslash G) = \overline{\bigoplus V}$. The Casimir operator becomes a constant multiplication in each V, so that $\Omega|_V = (1/4+\kappa^2)\cdot 1$. With $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, there are two possibilities: either $\kappa > 0$ or $i\kappa$ is equal to half an odd integer. According to the action of K, the space V is decomposed into K-irreducible subspaces: $V = \overline{\bigoplus_{p\in\mathbf{Z}} V_p}$, $\dim V_p \le 1$. If it is not trivial, V_p is spanned by a Γ-automorphic function on which the right translation by $\mathrm{k}[\theta]$ becomes the multiplication by the factor $e^{2ip\theta}$. It is called a weight vector.

Hereafter, we shall assume that V is in the unitary principal series, i.e., $\kappa > 0$. Then $\dim V_p = 1$ for all $p \in \mathbf{Z}$. Let φ_0 be a generator of V_0 that is of length one, i.e., a Maass wave. It is readily seen that V_p is generated by $\varphi_p = \mathbf{E}^p\varphi_0/\|\mathbf{E}^p\varphi_0\|$, where $\mathbf{E} = e^{2i\theta}(2iy\partial_x + 2y\partial_y - i\partial_\theta)$, and $\mathbf{E}^p = (\overline{\mathbf{E}})^{|p|}$ for $p<0$. The family $\{\varphi_p : p \in \mathbf{Z}\}$ is a complete orthonormal system in V. By definition we have the expansion

$$\varphi_p(\mathrm{g}) = \sum_{\mathbf{Z}\ni n\neq 0} W_n(\mathrm{g};\varphi_p). \tag{4}$$

The differential equation $\Omega\varphi_p = (1/4+\kappa^2)\varphi_p$, together with the side condition that $W_n(\mathrm{n}[x]\mathrm{gk}[\theta];\varphi_p) = e^{2pi\theta}e(nx)W_n(\mathrm{g};\varphi_p)$, and φ_p is cuspidal, implies that $W_n(\mathrm{a}[y];\varphi_p)$ is a constant multiple of the Whittaker function $W_{\lambda,\mu}$, with $\lambda = p$, $\mu = i\kappa$, which satisfies

$$\left[\left(\frac{d}{dy}\right)^2 - \frac{1}{4} + \frac{\lambda}{y} + \frac{1/4-\mu^2}{y^2}\right] W_{\lambda,\mu}(y) = 0. \tag{5}$$

Thus we may put

$$W_n(\mathrm{g};\varphi_p) = \frac{1}{2}(-1)^p|n|^{-1/2}\varrho_p(n) \cdot e^{2ip\theta}e(nx)\Gamma\left(\frac{1}{2}+i\kappa\right)\frac{W_{p\,\mathrm{sgn}(n),i\kappa}(4\pi|n|y)}{\Gamma(p\,\mathrm{sgn}(n)+1/2+i\kappa)} \tag{6}$$

with a constant $\varrho_p(n)$. Expressing the right side as a Jacquet transform, one may conclude that for all $p \in \mathbf{Z}$

$$\varrho_p(n) = \varrho_0(n) = \varrho_V(n), \tag{7}$$

say.

4. Fundamental assertions. We are now ready to give a brief yet rigorous proof of the unitaricity assertion that is mentioned above in relation with the Kirillov model.

Theorem 1. *For any integer $n \neq 0$ and any smooth vector $\varphi \in V$, we have*

$$\int_{\mathbf{R}^\times} |w_n(u;\varphi)|^2 d^\times u = \frac{\pi}{4}\frac{|\varrho_V(n)|^2}{|n|\cosh\pi\kappa}\|\varphi\|^2_{\Gamma\backslash G}.$$

Proof. Write

$$\varphi(\mathrm{g}) = \sum_{\mathbf{Z}\ni p} \alpha_p\varphi_p(\mathrm{g}), \tag{8}$$

with φ_p as above. Taking Fourier coefficients in θ on both sides, we see, via (4), that the assertion is equivalent to the orthogonality

$$\int_{\mathbf{R}^\times} w_n(u,\varphi_p)\overline{w_n(u,\varphi_q)}d^\times u = \delta_{p,q}\frac{\pi}{4}\frac{|\varrho_V(n)|^2}{|n|\cosh\pi\kappa},$$

with δ the Kronecker delta. In view of (6) and the fact that $W_{p,i\kappa}(y)$ is real, this is an elementary consequence of the following identity (see the formula 7.611(3) of [3]): For any α, β and for any $|\operatorname{Re}\mu| < 1/2$,

$$\int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y)\frac{dy}{y} = \frac{\pi}{(\alpha-\beta)\sin(2\pi\mu)} \cdot\left[\frac{1}{\Gamma(1/2-\alpha+\mu)\Gamma(1/2-\beta-\mu)} - \frac{1}{\Gamma(1/2-\alpha-\mu)\Gamma(1/2-\beta+\mu)}\right].$$

To prove this, we shall use the differential equation (5): We have

$$-\alpha\int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y)\frac{dy}{y} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \left[\left(\frac{d}{dy}\right)^2 - \frac{1}{4} + \frac{1/4-\mu^2}{y^2}\right] \cdot W_{\alpha,\mu}(y)W_{\beta,\mu}(y)\frac{dy}{y} \tag{9}$$

$$= \lim_{\varepsilon\to 0^+} \left[-W'_{\alpha,\mu}(\varepsilon)W_{\beta,\mu}(\varepsilon) + W_{\alpha,\mu}(\varepsilon)W'_{\beta,\mu}(\varepsilon)\right] - \beta\int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y)\frac{dy}{y}.$$

Computing the last limit with the Taylor expansion for Whittaker functions, we end the proof of the theorem. □

Next, we shall fix the kernel in (3), again with a brief yet rigorous argument. To this end, we shall first make some reductions: As is remarked below, it is enough to treat the case $\varphi = \varphi_p$ with an arbitrary p. One may assume that φ_0 is a simultaneous eigenvector of all Hecke operators, and in particular $\varrho_V(1) \neq 0$. Then we have that

$$\varrho_V(1)W_m(\mathrm{g};\varphi_p) = \varrho_V(m)e(mx)e^{2ip\theta}W_1(\mathrm{a}[|m|y];\varphi_{p\ \mathrm{sgn}(m)}),$$

and $\varrho_V(1)w_n(u;\varphi_p) = \varrho_V(n)w_1(nu;\varphi_p)$. Thus, in place of (3) with $\varphi = \varphi_p$, we may write

$$\varrho_V(n)W_m(\mathrm{g};\varphi_p) = \varrho_V(m)e(mx)e^{2ip\theta} \cdot \int_{\mathbf{R}^\times} j_V^{1,1}(\mathrm{sgn}(m)nu;\mathrm{a}[|m|y])w_n(u;\varphi_p)d^\times u.$$

Hence, it is enough to consider the situation $m = n = 1$ only. Let $G = B \sqcup N\mathrm{w}B$, $B = NA$, be the Bruhat decomposition. If $\mathrm{g} = \mathrm{n}[x]\mathrm{a}[y] \in B$, then one may put $j_V^{1,1}(u;\mathrm{g}) = e(x)\delta(u-y)u$ with the Dirac delta. If $\mathrm{g} = \mathrm{n}[x_1]\mathrm{w}\mathrm{n}[x_2]\mathrm{a}[y] \in N\mathrm{w}B$, then one may put $j_V^{1,1}(u;\mathrm{g}) = e(x_1+ux_2)j_V(u/y)$, with j_V given below. In this way we find that (3) is solved with the assertion:

Theorem 2. *Let* w *be the Weyl element. Then we have, for any smooth vector φ in V and $y > 0$,*

$$w_1(y;\varphi(*\cdot\mathrm{w})) = \int_{\mathbf{R}^\times} j_V(yu)w_1(u;\varphi)d^\times u,$$

where $j_V(u)$ is defined to be

$$\pi\frac{\sqrt{|u|}}{\sin\pi i\kappa}\left\{J_{-2i\kappa}^{\mathrm{sgn}(u)}(4\pi\sqrt{|u|}) - J_{2i\kappa}^{\mathrm{sgn}(u)}(4\pi\sqrt{|u|})\right\},$$

with $J_\nu^+ = J_\nu$ and $J_\nu^- = I_\nu$ in the usual notation for Bessel functions.

Proof. With (8) and elementary uniform bounds for the Bessel and Whittaker functions, one may conclude that it suffices to treat the case of weight vectors $\varphi = \varphi_p$. We then consider the Mellin transform:

$$\Gamma_p(s) = \int_0^\infty W_1(\mathrm{a}[y];\varphi_p)y^{s-3/2}dy,$$

which is regular for $\mathrm{Re}\, s > 0$. It is known that Γ_p continues meromorphically to $\mathbf{C}$, and satisfies

$$(10)\quad (-1)^p\Gamma_p(s) = 2^{1-2s}\pi^{-2s}\Gamma(s+i\kappa)\Gamma(s-i\kappa) \cdot (\cos\pi s\,\Gamma_p(1-s) + \cos\pi i\kappa\,\Gamma_{-p}(1-s)).$$

This is the local functional equation given in Theorem 5.15 of [4]; for the sake of completeness, we shall prove it directly later. Observe the L^2-Mellin pairs: For $0 < \mathrm{Re}\, s < 1/4$

$$(11)\quad 2^{1-2s}\pi^{-2s}\cos\pi s\,\Gamma(s+i\kappa)\Gamma(s-i\kappa) \longleftrightarrow -\frac{\pi}{\sin\pi i\kappa}\left(J_{2i\kappa}(4\pi\sqrt{y}) - J_{-2i\kappa}(4\pi\sqrt{y})\right);$$

for $0 < \mathrm{Re}\, s$

$$(12)\quad 2^{1-2s}\pi^{-2s}\cos\pi i\kappa\,\Gamma(s+i\kappa)\Gamma(s-i\kappa) \longleftrightarrow -\frac{\pi}{\sin\pi i\kappa}\left(I_{2i\kappa}(4\pi\sqrt{y}) - I_{-2i\kappa}(4\pi\sqrt{y})\right);$$

and $\Gamma_p(s) \leftrightarrow y^{-1/2}W_1(\mathrm{a}[y];\varphi_p)$ for $0 < \mathrm{Re}\, s$. We combine these three pairs with (10), and appeal to the Parseval formula for Mellin transforms. We are immediately led to the assertion of the theorem.

We need yet to prove (10): By (6) and a well-known integral representation of the Whittaker function via a Jacquet transform, we have

$$\Gamma_p(s) = \frac{1}{2}\pi^{-1/2-i\kappa}\varrho_V(1)\Gamma\left(\frac{1}{2}+i\kappa\right) \cdot \int_0^\infty y^{s-i\kappa-1}\int_{\mathcal{L}}\frac{e(y\xi)}{(\xi^2+1)^{1/2+i\kappa}}\left(\frac{\xi+i}{\xi-i}\right)^p d\xi dy.$$

Here $\mathrm{Im}\,\xi = 1/2$ for $\xi \in \mathcal{L}$. Assume temporarily that $0 < \mathrm{Re}\, i\kappa < 1/4 < \mathrm{Re}\, s < 1/2$. Exchange the order of integration, compute the resulting inner integral, and apply analytic continuation. We find that for $\kappa \in \mathbf{R}$, $0 < \mathrm{Re}\, s < 1$

$$\Gamma_p(s) = 2^{-s-1+i\kappa}\pi^{-1/2-s}\varrho_V(1)\Gamma\left(\frac{1}{2}+i\kappa\right) \cdot \Gamma(s-i\kappa)\left[\exp\left(-\frac{1}{2}\pi i(s-i\kappa)\right)\mathrm{L}_p(s) + \exp\left(\frac{1}{2}\pi i(s-i\kappa)\right)\mathrm{L}_{-p}(s)\right],$$

with

$$\mathrm{L}_p(s) = \int_0^\infty \frac{\xi^{-s+i\kappa}}{(\xi^2+1)^{1/2+i\kappa}}\left(\frac{\xi+i}{\xi-i}\right)^p d\xi.$$

Observe that the change of variable $\xi \to \xi^{-1}$ gives $\mathrm{L}_p(s) = (-1)^p\mathrm{L}_{-p}(1-s)$, which yields (10) under the present supposition on κ and s. Having this, replace the contour by the one, on which ξ starts at $+\infty$, goes down close to 0 along the positive real axis, encircles the origin once counterclockwise, and returns to $+\infty$, while $\arg \xi$ varies from 0 to 2π. We see that $\mathrm{L}_p(s)$ and hence $\Gamma_p(s)$ are meromorphic over $\mathbf{C}$. This ends the proof of Theorem 2. □

So far we have been working with V in the unitary principal series. In the case $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, there are no complementary series. We can, however, include such irreducible representations into our discussion, with a minor modification of the above argument. The same can be said about the discrete series representations. Further, the Eisenstein part is much analogous to the unitary principal series.

Here are some specific points to be commented in addition: As an application of Theorem 4.1 of [2], which corresponds to our Theorem 2 but lacks a rigorous proof, the authors show, in Chapter 8, that the Mellin transform of the Bessel kernel is the Gamma function of the relevant irreducible representation. In our proof of Theorem 2, the argument is reversed: it is obtained as a consequence of the local functional equation that we prove quickly from scratch. An advantage of our argument over [2], and over that on p. 454 of [9], too, is in that the pertinent convergence issues are easy to overcome, because the L^2-theory of Mellin transforms is available. At any events, the fundamental Theorem 2 is equivalent to Theorem 5.15 of Jacquet and Langlands [4], a fact which reveals the origin of the Bessel transform of Kuznetsov that once looked mysterious.

In an analogous way, the corresponding Bessel transform for the complex case that is given in Theorem 2 of [1] can be related with Theorem 6.4 of [4], as is to be shown in the next note of this series.

Concluding remark. In the light of what has been developed above, we shall give a tentative description of the nature of the spectral decomposition for the moment (1). This seems to suggest the existence of a unified theory of mean values of automorphic L-functions: Thus, let V be a cuspidal irreducible subspace of $L^2(\Gamma\backslash G)$. We may assume that all V are Hecke invariant. Let j_E stand for the limit as $\kappa \to 0$ in the definition of j_V. Note that j_E corresponds to $\zeta^2(s)$ the Hecke series arising from the central value of the modified Eisenstein series for $\mathrm{PSL}_2(\mathbf{Z})$, or more precisely $(\zeta(s)/\zeta(1-s))^2$ is the Mellin transform of $j_E(u)/(\pi\sqrt{|u|})$ over $\mathbf{R}^\times$. With this, put

$$\Xi_V(r) = \frac{1}{2}\int_{\mathbf{R}^\times} j_V\left(\frac{u}{r}\right) j_E(-u)\frac{d^\times u}{\sqrt{|u|}}. \tag{13}$$

Then, the cuspidal part of the spectral decomposition for (1) has the form

$$\sum_V \frac{|\varrho_V(1)|^2}{\cosh \pi\kappa} H_V\left(\frac{1}{2}\right)^3 \Theta_V(g). \tag{14}$$

Here H_V is the Hecke L-function attached to V, and

$$\Theta_V(g) = \int_0^\infty (r(r+1))^{-1/2} \cdot g_c\left(\log\left(1+\frac{1}{r}\right)\right)\Xi_V(r)dr, \tag{15}$$

with the cosine transform g_c, provided the weight g is real on $\mathbf{R}$. This follows from a combination of (11), (12) above and (4.4.16), (4.4.17), (4.7.4) of [6]. Our attention is at the construction of $\Theta_V(g)$. Since E stands here for the Eisenstein series and corresponds to $\zeta^2(s)$, one may wonder what the analogue of $\Theta_V(g)$ will be, if we consider, instead, the mean square of the Hecke series attached to a given cuspidal irreducible representation A. With A in the discrete series, the spectral decomposition of the mean square is obtained in [5], which can be readily transformed into an expression similar to (13)–(15). The resulting analogue of $\Xi_V(r)$ has an expression much alike as (13), which involves the factor $j_A(-u)$ in place of $j_E(-u)$ besides a minor complication. In particular, we may translate a phenomenon, observed in [5], into a structural statement. That is, the fact that there is no contribution by any V from the discrete series is equivalent to the trivial relation $j_V(u/r)j_A(-u) = 0$. On the other hand, if A is in the unitary principal series, it appears highly probable that the analogue of $\Theta_V(g)$ will again involve the function $j_V(u/r)j_A(-u)$. If this is indeed the case, it should lead us to a unified theory of mean values of automorphic L-functions.

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