On arithmetic infinite graphs

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(Communicated by Heisuke HIRONAKA, M. J. A., Feb. 14, 2000)

Abstract: We compute explicitly the Selberg trace formula for principal congruence subgroups Γ of $PGL(2, \mathbf{F}_q[t])$, which is the modular group in positive characteristic cases. It is known that $\Gamma \backslash X$ is an infinite Ramanujan diagram, where X is the $q+1$-regular tres. We express the Selberg zeta function for Γ as the determinant of the adjacency operator which is composed of both discrete and continuous spectra. They are rational functions in q^{-s}. We also discuss the limit distribution of eigenvalues of $\Gamma \backslash X$ as the level tends to infinity.

Key words: Function field; Selberg trace formula; Ihara-Selberg zeta function; Ramanujan graph/diagram; graph spectra.

1. Introduction. The Selberg trace formula and the Selberg zeta function have been well-explored originally for $PSL(2, \mathbf{R})$. Let Γ be a cofinite discrete subgroup of $PSL(2, \mathbf{R})$, which acts on the upper half-plane $\mathbf{H}$, then it is known that the Selberg zeta function $Z_\Gamma(s)$ attached to Γ is expressed as the determinant of the Laplacian. One of the purposes of this note is to give a new explicit example of the Selberg trace formula and the Selberg zeta function. We consider the case of function fields and treat principal congruence subgroups $\Gamma(A)(A \in \mathbf{F}_q[t])$ of $PGL(2, \mathbf{F}_q[t])$, which is an analog of $PSL(2, \mathbf{Z})$ in view of number theory. They act on the associated Bruhat-Tits tree X, so instead of non-compact arithmetic manifolds the infinite quotient graphs appear. In the case of finite graphs the trace formulas and Ihara-Selberg zeta functions are well investigated. Our results can be regarded as the first ones in the case of infinite graphs, which generalize those works.

Ramanujan graphs are defined as k-regular finite graphs whose nontrivial eigenvalues of adjacency operator have absolute values bounded by $2\sqrt{k-1}$. But it is a hard task to determine the eigenvalues for large graphs. The first construction of a family of Ramanujan graphs whose sizes tend to infinity are made by Lubotzky, Phillips, and Sarnak and independently Margulis. Morgenstern introduced the notation of Ramanujan diagrams, which is a generalization of Ramanujan graphs. In [M2] he showed the quotient graphs $\Gamma(A)\backslash X$ are Ramanujan diagrams by using the Ramanujan conjecture proved by Drinfeld. To date they are the only example of Ramanujan diagrams which are not finite graphs. In Section 4, we investigate the limit distribution of eigenvalues of these graphs $\Gamma(A)\backslash X$ when the level $\deg A$ tends to infinity.

In Section 2, we prepare some notations and preliminaries such as the Bruhat-Tits tree X and adjacency operator T on it. In Section 3, we give the explicit trace formula and the determinant expression of the Selberg zeta function. The detailed version will appear in forthcoming papers.

2. Preliminaries. Let $\mathbf{F}_q$ be the finite field with q elements and $\mathbf{F}_q[t]$ be the ring of polynomials in t over $\mathbf{F}_q$. Let k_∞ be the completion at infinity of the rational function field $k = \mathbf{F}_q(t)$ and r_∞ be the ring of its local integers. Then k_∞ is the field $\mathbf{F}_q((t^{-1}))$ of Laurent series in uniformizer t^{-1} over $\mathbf{F}_q$, and r_∞ is the ring $\mathbf{F}_q[[t^{-1}]]$ of Taylor series in t^{-1} over $\mathbf{F}_q$. If a element a in k_∞ is written as $\sum_{i=n}^{\infty} a_i t^{-i} (a_n \neq 0)$, then the norm $|a|_\infty$ of a is q^{-n}. Throughout we put $G = PGL(2, k_\infty)$ and $K = PGL(2, r_\infty)$. As is described in [Se1, II.1.1], we can endow G/K with the structure of the $q+1$ regular tree X. The tree X has a natural distance d, namely, if u and v are adjacent in X we let $d(u,v) = 1$. We write $V(X)$ for the set of vertices of this tree X, i.e., cosets of G/K, and $E(X)$ for the set of edges of X. The group G acts on the tree X as a group of automorphisms. This action of G on X can be extended to the boundary ∂X of X, the

1991 Mathematics Subject Classification. Primary 11M36, 11F72, Secondary 05C50, 58J50.

element of which is a equivalence class of half-lines with two half-lines being equivalent if they differ at most in a finite graph.

Let Γ be a subgroup of G which acts without inversions on X, then it naturally gives rise to a quotient graph $\Gamma\backslash X$. If Γ is a discrete subgroup of G of finite covolume, Lubotzky [Lu, Theorem 6.1.] shows that the quotient graph $\Gamma\backslash X$ is the union of a finite graph $\mathcal{F}_0$ together with finitely many infinite half lines. For example, when $\Gamma(1) := PGL(2, \mathbf{F}_q[t])$ the quotient graph $\Gamma(1)\backslash X$ is isomorphic to a half line [Se1, II.1.6]. The quotient graph $\Gamma\backslash X$ can be made into an atomic measure space induced from a Haar measure of G, which we normalize so that the volume of K is 1. Denote the stabilizer of $v \in V(\Gamma\backslash X)$ and $e \in E(\Gamma\backslash X)$ in Γ by Γ_v and Γ_e respectively. Then we see that a vertex $v \in V(\Gamma\backslash X)$ have the measure $m(v) = |\Gamma_v|^{-1}$ (see [Se1, II.1.5]). For later use we put $m(e) = |\Gamma_e|^{-1}$, where $e \in E(\Gamma\backslash X)$. In this paper we consider $\mathbf{C}$-valued functions defined on vertices.

Now we define a natural operator on X, which we call the adjacency operator, by

$$(1)\quad (Tf)(v) := \sum_{d(v,u)=1} f(u) \quad (f\colon V(X) \to \mathbf{C}).$$

It induces an operator on functions which satisfies $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$ and $g \in V(X)$, that is, just functions on the quotient graph $\Gamma\backslash X$. The induced operator can be represented as

$$(Tf)(v) = \sum_{e=(v,u)\in E(\Gamma\backslash X)} \frac{m(e)}{m(v)} f(u) \qquad (f\colon V(\Gamma\backslash X) \to \mathbf{C}).$$

More generally, we define the operator $T_m (m = 0, 1, 2, \ldots)$, which average functions on $V(X)$ at distance m:

$$(2)\ (T_m f)(v) := \sum_{d(v,u)=m} f(u) \quad (f\colon V(X) \to \mathbf{C}).$$

Then it can be seen that the following recursive relations hold:

$$T_1^2 = T_2 + (q+1)T_0$$
$$T_1 T_m = T_{m+1} + qT_{m-1} \quad (m \geq 2).$$

These relations yield the following identity:

$$(3)\qquad \sum_{m=0}^{\infty} T_m u^m = \frac{1-u^2}{1 - T_1 u + qu^2},$$

where u is an indeterminate.

3. Trace formulas and Selberg zeta functions. In the following, we let q be an odd prime power and Γ be a principal congruence subgroup of G:

$$\Gamma(A) = \{\gamma \in PGL(2, \mathbf{F}_q[t]) | \gamma \equiv I \pmod{A}\} \qquad (A \in \mathbf{F}_q[t]).$$

We also assume $\deg A = a \geq 1$. Then the quotient graph $\Gamma\backslash X$ is an infinite graph, so that there will be continuous spectra as well as discrete spectra of T. We will define the Eisenstein series for each cusp. Here let $\kappa_1, \ldots, \kappa_\mu$ be a complete set of inequivalent cusps for Γ. Let Γ_{κ_i} be the stabilizer in Γ of κ_i and take an element $\tilde{\kappa_i} \in G$ such that $\tilde{\kappa_i}\infty = \kappa_i$. Then the Eisenstein series for κ_i is defined by

$$E_i(g,s) := \sum_{\gamma \in \Gamma_{\kappa_i}\backslash\Gamma} \psi_s(\tilde{\kappa}_i^{-1}\gamma g) \qquad (g \in G/K, \mathrm{Re}(s) > 1),$$

where $\psi_s(g) := |\det g|_\infty^s h((0\,1)g)^{-2s}$ and $h((x\,y)) := \sup\{|x|_\infty, |y|_\infty\}$.

The Eisenstein series $E_i(g,s)$ is invariant under Γ, so it can be expanded as a Fourier series at each cusp κ_j. In the case of principal congruence groups, Li [L1] obtains an explicit form of the Fourier series in terms of the L-functions associated to the characters χ on $\mathbf{F}_q[t] \mod A$. The constant terms of the Fourier series of $E_i(g,s)$ at cusps κ_j define the $\mu \times \mu$ matrix $\Phi(s)$ which is called the scattering matrix of Γ. Its determinant $\varphi(s) := \det \Phi(s)$ is called the scattering determinant of Γ. Then $\Phi(s)$ satisfies the functional equation $\Phi(s) = \Phi(1-s)$. By the above computations of [L1] we see that $\varphi(s)$ is a rational function in q^{2s}, so we put

$$(1)\qquad \varphi(s) = c\frac{(q^{2s} - qa_1)\cdots(q^{2s} - qa_m)}{(q^{2s} - qb_1)\cdots(q^{2s} - qb_n)},$$

where c, a_j, b_j are constants and we assume that the right hand side is written to be irreducible. We find that the continuous spectrum are furnished by Eisenstein series for each cusp and are parametrized by the interval $[-2\sqrt{q}, 2\sqrt{q}]$. The explicit trace formula for Γ can be found in [N1].

Theorem 3.1 [N1]**.** Let $\mathfrak{P}_\Gamma$ denote the set of primitive hyperbolic conjugacy classes of Γ and M be the number of discrete spectra of T for $\Gamma\backslash X$. For $\{P\} \in \mathfrak{P}_\Gamma$, we put $N(P) = \sup\{|\lambda_i|_\infty^2 \mid \lambda_i$ is an eigenvalue of the matrix $P\}$ and let $\deg P := \log_q N(P)$. Assume that the sequence $c(n) \in \mathbf{C} (n \in$

$\mathbf{Z}$) satisfies $c(n) = c(-n)$ and $\sum_{n\in\mathbf{Z}} q^{|n|/2}|c(n)| < \infty$. Then we have the following formula:

$$\begin{aligned}(2) \qquad & \sum_{n=1}^{M} h(r_n) \\ = \ & \mathrm{vol}(\Gamma\backslash X)k(0) \\ & + \sum_{\{P\}\in\mathfrak{P}_\Gamma} \sum_{l=1}^{\infty} \frac{\deg P}{q^{(l\deg P)/2}} c(l \deg P) \\ & + \left(\mu - \mathrm{Tr}\Phi\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}c(0) + \sum_{m=1}^{\infty} c(2m)\right) \\ & + \frac{1}{4\pi}\int_{-\pi/\log q}^{\pi/\log q} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr \\ & -\mu\left(a + \frac{1}{q-1}\right)c(0).\end{aligned}$$

Here the functions $h(\cdot), k(\cdot)$ are determined by $c(\cdot)$ via the Selberg transform.

We now turn to investigate the Selberg zeta function for Γ, which is defined by

$$(3) \qquad Z_\Gamma(s) := \prod_{\{P\}_\Gamma\in\mathfrak{P}_\Gamma} \left(1 - N(P)^{-s}\right)^{-1}.$$

We use the trace formula (2) by plugging the following test function $c(n)$

$$c(n) = \begin{cases} -(\log q)q^{-|n|(s-1/2)} & n \neq 0 \\ 0 & n = 0, \end{cases}$$

where $s \in \mathbf{C}$ is fixed with $\mathrm{Re}(s) > 1$. Now we define our determinant of T associated to $\Gamma\backslash X$ by

$$\det(T, s) := \det{}_D(T, s) \cdot \det{}_C(T, s),$$

where

$$\begin{aligned}\det{}_D(T, s) &:= \det{}_D(1 - Tq^{-s} + q^{1-2s}) \\ &= \prod_{n=1}^{M}(1 - \lambda_n q^{-s} + q^{1-2s}), \\ \det{}_C(T, s) &:= \prod_{|b_j|<1}(1 - q^{-2s+1}b_j) \\ & \quad \cdot \prod_{|b_j|>1}(1 - q^{-2s+1}b_j^{-1})^{-1},\end{aligned}$$

and b_j is given by (1). Then we have the next result.

Theorem 3.2 [N1]. The Selberg zeta function $Z_\Gamma(s)$ attached to Γ can be expressed by the determinant of T:

$$(4) \quad Z_\Gamma(s)^{-1} = (1 - q^{-2s})^{\chi}(1 - q^{-2s+1})^{-\rho}\det(T, s),$$

where $\chi := \mathrm{vol}(\Gamma\backslash X)(q-1)/2$, $\rho := (1/2)\mathrm{Tr}(I_\mu - \Phi(1/2))$ and I_μ is the $\mu \times \mu$-identity matrix.

4. The distribution of eigenvalues. As described in Introduction, by using the Ramanujan conjecture proved by Drinfeld, Morgenstern [M2] shows that any nontrivial discrete spectrum λ of T on $\Gamma\backslash X$ satisfies $|\lambda| \leq 2\sqrt{q}$. We put a normalized operator $T' = T/\sqrt{q}$ and let D' be the set of the nontrivial discrete spectra of T'. Every element λ' of D' satisfies $|\lambda'| \leq 2$. Then we consider certain limit distribution of eigenvalues of T' on $\Gamma\backslash X$.

First we prepare two probability measures on $\Omega = [-2, 2]$. One is the Sato-Tate measure or Wigner semi-circle:

$$\mu_\infty(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx.$$

The other is defined for a real number $q(> 1)$ by

$$\mu_q(x) = \frac{q+1}{(q^{1/2} + q^{-1/2})^2 - x^2}\mu_\infty(x).$$

The Chebychev polynomials of the second kind $X_m(x)$ $(m = 0, 1, 2, \ldots)$ are known to be orthogonal with respect to $\mu_\infty(x)$ and they satisfy

$$\sum_{m=0}^{\infty} X_m(x)u^m = \frac{1}{1 - xu + u^2},$$

where u is an indeterminate. Next we define the polynomials $X_{m,q}(x)(m = 0, 1, 2, \ldots)$ by

$$X_{m,q}(x) := X_m(x) - q^{-1}X_{m-2}(x),$$

where we let $X_m(x) := 0$ for $m < 0$. Then we have

$$(1) \qquad \int_\Omega X_{m,q}(x)d\mu_q(x) = \begin{cases} 1 \ (m = 0) \\ 0 \ (m > 0) \end{cases}$$

and

$$(2) \qquad \sum_{m=0}^{\infty} X_{m,q}(x)u^m = \frac{1 - u^2/q}{1 - xu + u^2}.$$

Now we normalize the operator T_m in (2) as $T'_m = T_m/q^{m/2}$. Then from (3) we have

$$(3) \qquad \sum_{m=0}^{\infty} T'_m u^m = \frac{1 - u^2/q}{1 - T'u + u^2}.$$

Hence (2) and (3) yield

$$(4) \qquad T'_m = X_{m,q}(T').$$

After these preparations, we can describe the following result. For the details of the proof and other investigations, see [N2].

Theorem 4.1 [N2]**.** Let an odd prime power q be fixed. Then for any $\{A_i\}(i = 0, 1, 2, \dots; A_i \in \mathbf{F}_q[t])$ such that $\deg A_i \to \infty$ as $i \to \infty$, the discrete spectra $D_i^{'}$ of $T^{'} = T/\sqrt{q}$ on $\Gamma(A_i)\backslash X$ are equidistributed with respect to the measure $\mu_q(x)$ on $\Omega = [-2, 2]$. That is, let $C(\Omega)$ be the space of $\mathbf{R}$-valued continuous functions for Ω, then for any $f(x) \in C(\Omega)$ the following holds:

$$\lim_{i\to\infty} \frac{1}{|D_i^{'}|} \sum_{\lambda^{'} \in D_i^{'}} f(\lambda^{'}) = \int_{\Omega} f(x) d\mu_q(x). \tag{5}$$

Sketch of the proof. The space spanned by the set of polynomials $\{X_{m,q}\}(m = 0, 1, 2, \dots)$ is dense in $C(\Omega)$, so it suffices to check that $f = X_{m,q}$ satisfies (5) for each m. We write $\mathrm{Tr}X_{m,q}(T^{'})$ the sum of the discrete spectra of T_m' for $\Gamma(A) \setminus X$, then by (4) we have $\mathrm{Tr}X_{m,q}(T^{'}) = \mathrm{Tr}T_m^{'}$. Let N_m be defined by

$$N_m := \sum_{\substack{\deg P | m \\ P \in \mathfrak{P}_\Gamma}} \deg P,$$

then the Selberg zeta function (3) can be described as

$$Z_\Gamma(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m \right), \tag{6}$$

where $u = q^{-s}$. Now taking the logarithmic derivative of (4) and (6) in u, by (3) we have the formula which represents $\mathrm{Tr}T_m$ in terms of $\{N_m\}$.

This formula contains the terms of the contribution of the scattering determinant $\varphi(s)$. Taking into account that poles of the scattering determinant $\varphi(s)$ correspond to zeros of certain product of Dirichlet L-functions mod A, it is possible to estimate these terms. When $\deg A \to \infty$ we have $N_m \to 0$ for each m and by the trace formula (2) we have $|D^{'}| \sim \mathrm{vol}(\Gamma(A)\backslash X)$. Combing the above facts, as $\deg A \to \infty$, $(1/|D^{'}|)\mathrm{Tr}T_m^{'}$ tends to zero for each $m(\neq 0)$, and hence by (1) we have the assertion. □

Acknowledgements. The author would like to express his sincere appreciation to Prof. S. Koyama and Prof. N. Kurokawa for discussions and encouragement.

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