A note on the mean value of the zeta and L-functions. IX

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The aim of the present note is to indicate the possibility of inverting our explicit formula [6, Theorem 4.2] for the fourth power moment of the Riemann zeta-function:

$$\mathcal{Z}_2(g) = \int_{-\infty}^{\infty} \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4 g(t)dt$$

where the weight g is to satisfy certain decay conditions.

To make this notion explicit we need first to introduce basic concepts from the theory of automorphic forms over the full modular group $\Gamma = PSL(2, \mathbf{Z})$ (see [6, Chapters 1–3]). Thus, let $\{1/4+\kappa_j^2\}$ ($\kappa_j>0; j=1,2,\ldots$) be the discrete spectrum of the hyperbolic Laplacian $\Delta = -y^2((\partial/\partial x)^2 + (\partial/\partial y)^2)$ acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure. Let $\{\psi_j\}$ be a maximal orthonormal system such that $\Delta\psi_j = (1/4+\kappa_j^2)\psi_j$ for each $j\ge 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n\ge 1$, where

$$(T(n)f)(z) = n^{-1/2}\sum_{ad=n}\sum_{b=1}^{d} f\Big(\frac{az+b}{d}\Big)$$

is the Hecke operator. We may further assume that $\psi_j(-\bar{z}) = \epsilon_j\psi_j$ with $\epsilon_j=\pm1$. We then define

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n)n^{-s},$$

which we call the Hecke series associated with ψ_j, and which can be continued to an entire function.

The integral $\mathcal{Z}_2(g)$ is expanded over the spectra of Δ, yielding the explicit formula under present consideration. Its part $\mathcal{Z}_{2,d}(g)$ corresponding to the discrete spectrum has the following expression:

$$\mathcal{Z}_{2,d}(g) = \sum_{j=1}^{\infty}\alpha_j H_j\left(\frac{1}{2}\right)^3 \Lambda(\kappa_j;g), \tag{1}$$

where $\alpha_j = |\rho_j|^2/\cosh(\pi\kappa_j)$ with the first Fourier coefficient ρ_j of ψ_j, and

$$\begin{aligned}\Lambda(r;g) = \mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big)&\\ \times\int_0^{\infty}(y(1+y))^{-1/2}g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)&\frac{\Gamma(1/2+ir)^2}{\Gamma(1+2ir)}\\ \times F\Big(\frac{1}{2}+ir,\frac{1}{2}+ir;1+2ir;-\frac{1}{y}\Big)&y^{-1/2-ir}dy\Big]\end{aligned} \tag{2}$$

with

$$g_c(x) = \int_{-\infty}^{\infty} g(t)\cos(xt)dt,$$

and F the hypergeometric function.

We regard $\Lambda(r;g)$ as an integral transform of g. Then we claim that for a given h satisfying appropriate conditions one may find a g such that

$$\Lambda(r;g) = \mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big)h(r)\Big]. \tag{3}$$

Since other parts of the explicit formula for $\mathcal{Z}_2(g)$ have structures essentially the same as (1), this solution g gives rise to the notion stated at the beginning.

We shall thus solve the integral equation (3). Our discussion will, however, be formal in the sense that convergence issues and necessary estimations are all skipped. This is to show the core of our idea, which is in fact very simple: We begin with the observation that the expression (2) can be written as

$$\begin{aligned}\Lambda(r;g) = 2\mathrm{Re}\Big[\Big(1+\frac{i}{\sinh(\pi r)}\Big)\int_0^{\infty}(y(1+y))^{-1/2}&\\ \times g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)Q_{-1/2+ir}(1+2y)dy\Big]&,\end{aligned}$$

where $Q_\nu(z)$ is the Legendre function of order ν of the second kind. We have used the relation

$$\begin{aligned}Q_\nu(z) = &\frac{\sqrt{\pi}\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)}\\ &\times(z-1)^{-\nu-1}F\Big(\nu+1,\nu+1;2\nu+2;\frac{2}{1-z}\Big),\end{aligned}$$

which holds, say, for $z>1$ and $\nu\ne -1,-2,\ldots$ (via analytic continuation applied to the first formula in the problem 4 on [3, p. 200]). Namely, the equation

(3) may be put as

$$h(r) = 2\int_0^\infty (y(1+y))^{-1/2} \times g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)Q_{-1/2+ir}(1+2y)dy.$$

By the functional equation

$$\sin(\pi\nu)\{Q_\nu(z) - Q_{-\nu-1}(z)\} = \pi\cos(\pi\nu)P_\nu(z) \tag{4}$$

(see [3, (7.5.2)]) we have, further,

$$\begin{aligned} i\frac{h(r)-h(-r)}{\pi\tanh(\pi r)} &= 2\int_0^\infty (y(1+y))^{-1/2} \\ &\quad\times g_c\Big(\log\Big(1+\frac{1}{y}\Big)\Big)P_{-1/2+ir}(1+2y)dy \\ &= 2\int_1^\infty (y^2-1)^{-1/2} \\ &\quad\times g_c\left(\log\frac{y+1}{y-1}\right)P_{-1/2+ir}(y)dy, \end{aligned} \tag{5}$$

where $P_\nu(z)$ is the Legendre function of order ν of the first kind. Then we invoke the Mehler–Fock inversion formula (see [3, p. 221]):

$$f(x) = \int_0^\infty r\tanh(\pi r)P_{-1/2+ir}(x) \times \int_1^\infty f(y)P_{-1/2+ir}(y)dydr, \tag{6}$$

which holds for f satisfying a non-stringent condition. Hence we have, for $x > 1$,

$$g_c\left(\log\frac{x+1}{x-1}\right) = \frac{i}{2\pi}(x^2-1)^{1/2}\int_{-\infty}^\infty rh(r)P_{-1/2+ir}(x)dr,$$

where we have used the relation

$$P_{-1/2+ir}(x) = P_{-1/2-ir}(x),$$

a consequence of (4). That is, we have, for $\xi > 0$,

$$g_c(\xi) = \frac{i}{2\pi\sinh(\xi/2)} \times \int_{-\infty}^\infty rh(r)P_{-1/2+ir}\Big(\coth\frac{\xi}{2}\Big)dr. \tag{7}$$

From this we can recover g via the inverse Fourier-cosine transform. This ends our formal discussion. It is just a matter of housekeeping to see whether the g given by (7) satisfies the condition that is required in applying (6) to (5).

In particular our explicit formula for $\mathcal{Z}_2(g)$ can thus be regarded as a consequence of geometrical symmetries of the hyperbolic plane. That which controls the symmetries is the spherical harmonics, which is in fact closely related to the free space resolvent kernel of Δ (see [6, p. 179]). The structural nature of this fact induces the speculation that analogous relations should occur in more general geometrical settings such as those generated by Lie groups of real rank one, because our formula for $\mathcal{Z}_2(g)$ is actually an elaborated transformation of the trace formula of the Kuznetsov type for the group Γ; and as Miatello and Wallach [4] showed there are counterparts of the trace formula for discrete subgroups of those Lie groups (see also our [7]). We are not yet in a position to say anything explicit about our speculation. We have, however, made already some experiments ([7]–[9]) with the Picard group $PSL(2, \mathbf{Z}[i])$, and got evidence supporting our supposition. Also, turning to a different direction, we can probably discuss, along our present context, mean values of individual Hecke series associated with cusp-forms either holomorphic or real-analytic (see Jutila [2] and our [5]). This possibility appears to have relevance to the non-vanishing hypothesis occurring in [2] and [5] (see also the concluding remark below). These ramifications as well as digressions to binary additive divisor problems, which have intrinsic relations with mean value problems of L-functions, will be discussed with rigorous arguments elsewhere.

Concluding remark. Recently Ivić [1] made a remarkable progress in the theory of Hecke series by establishing the bound

$$\sum_{K\le\kappa_j\le K+1} \alpha_j H_j\left(\frac{1}{2}\right)^3 \ll K^{1+\varepsilon} \tag{8}$$

for any $K \ge 1$ and $\varepsilon > 0$. For this purpose he enhanced, in an ingenious way, the efficiency of our argument with which we had obtained an asymptotic formula for the sum

$$\sum_{\kappa_j\le K} \alpha_j H_j\left(\frac{1}{2}\right)^3,$$

and thus proved the non-vanishing lemma: for infinitely many κ's

$$\sum_{\kappa_j=\kappa} \alpha_j H_j\left(\frac{1}{2}\right)^3 \ne 0$$

(see [6, Theorem 3.2]). The above observation, which was in fact made several years ago but has been left unpublished because of no particular idea of applications, has an obvious relevance to the advance made by Ivić, for we now have an explicit formula for the sum

$$\mathrm{Re} \sum_{j=1}^{\infty} \alpha_j H_j \left(\frac{1}{2}\right)^3 \left(1 + \frac{i}{\sinh(\pi\kappa_j)}\right) h(\kappa_j)$$

with a given weight h. Though we have yet to make explicit the condition that h has to satisfy, our new formula provides a structural instrument to understand the bound (8); and it now appears reasonable to us to conjecture that there should be a way to relate the bound for $\zeta(1/2 + it)$ with that for $H_j(1/2)$.

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