On the Rank of Elliptic Curves with Three Rational Points of Order 2. II

By Shoichi KIHARA

Department of Neuropsychiatry School of Medicine Tokushima University

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In this note, we prove.

Theorem. *There are infinitely many elliptic curves with rank* ≥ 5 *over* $\boldsymbol{Q}$, *which have* 3 *distinct non-trivial rational points of order* 2.

This improves the result of our previous paper [2], where we proved the theorem just as above with rank "≥ 4", however, instead of "≥ 5".

To prove our Theorem, we shall follow the same method as in [2], and use in particular the Proposition 1 in that paper. Moreover, we shall utilize an auxiliary elliptic curve C with positive rank as in [3].

1. As in [2], let $K = \boldsymbol{Q}(t)$, t being a variable, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3 + 15t, 5 + 9t, 9 + 5t, 45 + t)$, and $\beta = 45t$, then we obtain the following elliptic curve

$$\varepsilon \qquad y^2 = A_0x^4 + B_0x^2 + C_0,$$

where $A_0 = 3136(3t^2 - 35)(5t^2 - 37)(15t^2 + 241)$,

$$B_0 = -6272(184725t^6 - 4373183t^4 + 25324735t^2 - 32932757),$$

$$C_0 = (45t)^2A_0.$$

Then ε has the following 5 K points:

$$P_0 = (3, -168(225t^4 - 1154t^2 - 8287)),$$
$$P_1 = (-3, 168(225t^4 - 1154t^2 - 8287)),$$
$$P_2 = (5, 280(135t^4 - 1550t^2 + 8583)),$$
$$P_3 = (9, 504(75t^4 - 2454t^2 + 9547)),$$
$$P_4 = (45, 2520(15t^4 + 2850t^2 - 26417)).$$

As A_0, B_0, and C_0 satisfy the conditions for A, B, and C in Proposition 1 in [2] and $P_0 \in \varepsilon$, ε has 3 distinct, non-trivial K-points of order 2.

2. Next, let us consider the following elliptic curve:

$$C : q^2 = p(p^2 - 20406000p + 77192390246400).$$

(4907760, 2355724800) is on C, and by Lutz- Nagell theorem, this point is of infinite order in the Mordell-Weil group of C, so that C has positive rank.

Let $\boldsymbol{Q}(C)$ be the function field of C. We consider ε over $\boldsymbol{Q}(C)$, like in [3], by specializing $t = q/(420p)$.

Then we have the point $P_5 = (x_5, y_5)$ on ε, where

$$x_5 = (-31p + 149360640)/(p - 8785920),$$

$$y_5 = (-157057064941217386095443548569600000 + 1361027170915054805833482240000p - 4110390293001362472960 0p^2 + 5132010223042560p^3 - 235101184p^4 + 3p^5)/(2469600p^2(p - 8785920)).$$

Proposition. $\boldsymbol{Q}(C)$ -rank of ε is at least 5.

Proof. Let ψ_{p_0} be the birational transformation defined in [2] and $Q_i = \psi_{p_0}(P_i)$, $i = 1, \ldots, 5$.

Specializing $(p, q) = (4907760, 2355724800)$, we have 5 rational points $R_1, \ldots, R_5$ obtained from $Q_1, \ldots, Q_5$.

By using calculation system PARI, we see that the determinant of the matrix $(< R_i, R_j >)$ $(1 \leq i, j \leq 5)$ associated to the canonical height is 12244.17. Since this determinant is non-zero, we see that $R_1, \ldots, R_5$ are independent points.

So we see $Q_1, \ldots, Q_5$ are independent. Q.E.D.

Now this Proposition and Theorem 20.3 in [1] establishes our Theorem.

References

[1] J. H. Silverman: The arithmetic of elliptic curves. Graduate Texts in Math., vol. 106, Springer-Verlag, New York (1986).

[2] S. Kihara: On the rank of elliptic curves with three rational points of order 2. Proc. Japan Acad., **73A**, 77–78 (1997).

[3] S. Kihara: On the rank of the elliptic curve $y^2 = x^3 + k$. II. Proc. Japan Acad., **72A**, 228–229 (1996).