Armendariz Rings

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(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1997)

1. Introduction. Let R be a domain (commutative or not) and $R[x]$ its polynomial ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. (This notation for the coefficients of $f(x)$ and $g(x)$ will be followed in the absence of explicit mention.) It is an elementary exercise to prove that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j, since either $f(x) = 0$ or $g(x) = 0$. (Of course the converse always holds.)

E. Armendariz ([1], Lemma 1) noted that the above result can be extended to the class of reduced rings, i.e., rings without non-zero nilpotent elements. In order to study additional classes of rings having this property we introduce the following definition.

1.1. Definition. A ring R is said to have the Armendariz property (or is an *Armendariz ring*) if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)\,g(x) = 0$, we have $a_i b_j = 0$ for every i and j.

By a ring we mean an associative ring with identity. However, the assumption of the existence of identity can be omitted in many places. Many remarks are thus valid in the context of "rings" and subrings (i.e., subrings which may not inherit the identity of the over-ring). For defining left/right zero-divisors, we shall refer to ([4], p. 88).

In addition to reduced rings, there are large classes of rings which are Armendariz. If R is a commutative P.I.D and A an ideal of R, then R/A is Armendariz (Theorem 2.2). If K is a field and V is a vector space over K, then the ring K $(+)$ V (see 1.2 for notation) is an Armendariz ring (Corollary 2.9).

For constructing examples of both Armendariz rings and non-Armendariz rings, we shall use the following principle of idealisation due to Nagata ([6], p.2).

1.2. Let R be a commutative ring and M an R-module. The R-module $R \oplus M$ acquires a ring structure where the product is defined by

$$(a, m)(b, n) = (ab, an + bm).$$

We shall use the notation $R\ (+)\ M$ for this ring. If M is not zero, this ring is not reduced, since M can be identified with the ideal $0 \oplus M$ which has square zero. (It seems appropriate to call this ring as "R Nagata M").

We shall also need the following variants of the construction in 1.2.

1.3. Let R be a commutative ring and $h: R \to R$ a ring homomorphism. Let M be an R-module. On modifying the definition in 1.2 to

$$(a, m)(b, n) = (ab, h(a)n + bm),$$

we get a (non-commutative) ring structure on $R \oplus M$ which we shall denote by $R\ (+)_h M$.

1.4. Let R be a ring and A an ideal of R. The factor ring $\bar{R} = R/A$ has the natural structure of a left R-, right R- bimodule. Denote $\bar{a} = a + A \in \bar{R}$ for each $a \in R$. We use this structure to define a ring structure on $R \oplus (R/A)$ as follows:

$$(r, \bar{a})(r', \bar{a}') = (rr', \overline{ra' + ar'}).$$

We denote this ring by $R\ (+)\ R/A$. Its properties are similar to those of $R\ (+)\ M$.

2. Rings which have the Armendariz property. It is easy to see that subrings of Armendariz rings are also Armendariz. However, factor rings need not be so (see 3.3). If $\{R_i\}_{i \in I}$ are Armendariz, so is ΠR_i. We begin with examples of familiar non-reduced rings which are Armendariz.

2.1. Proposition. For each integer n, $\boldsymbol{Z}/n\boldsymbol{Z}$ is an Armendariz ring, which is not reduced whenever n is a natural number which is not square free.

Proof. We first consider the case $n = p^m$, p a prime. Denote by $\overline{f(x)}$, $\overline{g(x)}$ the cosets of $f(x)$, $g(x)$ (mod $p^m \boldsymbol{Z}[x]$), respectively. Assume $\overline{f(x)g(x)} = 0$, i.e. $p^m \mid f(x)g(x)$. Since p is a prime, it follows that $f(x) = p^r f'(x)$ and $g(x) = p^s g'(x)$ for some f' and g' satisfying the conditions that the $g.\ c.\ d.$ of the coefficients of f' (also of g') is not divisible by p. Clearly $r+s \geq m$. It follows that $\bar{a}_i\bar{b}_j = 0$ for every i and j, showing that $\boldsymbol{Z}/p^m\boldsymbol{Z}$ is Armendariz.

Let n be a natural number. Then $n = p_1^{e_1}p_2^{e_2} \cdots p_i^{e_i}$ where p_k's are primes. By the Chinese remainder theorem,

$$\boldsymbol{Z}/n\boldsymbol{Z} \cong \boldsymbol{Z}/p_1^{e_1}\boldsymbol{Z} \oplus \boldsymbol{Z}/p_2^{e_2}\boldsymbol{Z} \oplus \dots \oplus \boldsymbol{Z}/p_i^{e_i}\boldsymbol{Z}.$$

Since each $\boldsymbol{Z}/p_k^{e_k}\boldsymbol{Z}$ is Armendariz, it follows that $\boldsymbol{Z}/n\boldsymbol{Z}$ is Armendariz.

The following generalisation of 2.1 has a similar proof.

2.2. Theorem. If R is a commutative P.I.D and A an ideal of R, then R/A is Armendariz.

2.3. Theorem. Let R be a domain, A an ideal of R. Suppose R/A is Armendariz. Then R $(+)$ R/A is Armendariz. (See 1.4 for definition of R $(+)$ R/A.)

Proof. Let $f(x)$, $g(x)$ be elements of $\{R$ $(+)$ $R/A\}[x]$, where

$$f(x) = \sum_{i=0}^{m} (a_i, \bar{u}_i)x^i = (f_0(x), \overline{f_1(x)}) \quad \text{and}$$
$$g(x) = \sum_{j=0}^{n} (b_j, \bar{v}_j)x^j = (g_0(x), \overline{g_1(x)}).$$

If $f(x)g(x) = 0$, we have $(f_0(x), \overline{f_1(x)})(g_0(x), \overline{g_1(x)}) = 0$. Thus we have the following equations:

$$\begin{cases} f_0(x)g_0(x) = 0 & \text{(I)} \\ \overline{f_0(x)g_1(x) + f_1(x)g_0(x)} = 0 & \text{(II)} \end{cases}$$

Case 1. $f_0(x) = 0$. Then (II) becomes $\overline{f_1(x)g_0(x)} = 0$ over R/A. Since R/A is Armendariz, it follows that $\overline{u_ib_j} = 0$ for eyery i and j. Also $f_0(x) = 0$ implies that $a_i = 0$ for all i. We conclude that $(a_i, \bar{u}_i)(b_j, \bar{v}_j) = (a_ib_j, \overline{a_iv_j + u_ib_j}) = 0$ for every i and j.

Case 2. $g_0(x) = 0$. This case is similar to case 1.

As a special case of the above proposition, we have the following corollary.

2.4. Corollary. $\boldsymbol{Z}$ $(+)$ $\boldsymbol{Z}/n\boldsymbol{Z}$ is Armendariz for each integer n.

It follows from 2.3 that if R is a domain then R $(+)$ R is Armendariz. This result can be extended to reduced rings. The following properties of these rings will be used: *i*) If a, b are elements of a reduced ring then $ab = 0$ if and only if $ba = 0$. ii) Reduced rings are Armendariz. iii) If R is reduced, then so is the ring $R[x]$. We shall also identify $\{R$ $(+)$ $R\}[x]$ with the ring $R[x]$ $(+)$ $R[x]$ in a natural manner.

2.5. Proposition. Let R be a reduced ring. Then the ring R $(+)$ R is Armendariz.

Proof. Let $f(x) = (f_0(x), f_1(x))$, $g(x) = (g_0(x), g_1(x))$ be elements of $\{R$ $(+)$ $R\}[x]$ satisfying $f(x)g(x) = 0$.

Write $f(x) = \sum_{i=0}^{m} (a_i, u_i)x^i$, and $g(x) = \sum_{j=0}^{n} (b_j, v_j)x^j$, with corresponding representations for $f_k(x)$, $g_k(x)$ (for $k = 0,1$).

Now we have

(A) $f_0(x)g_0(x) = 0$.

(B) $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$.

Since $R[x]$ is reduced, (A) implies

(C) $g_0(x)f_0(x) = 0$.

Multiplying equation (B) by $g_0(x)$ on the left and using (C) we get $g_0(x)f_1(x)g_0(x) = 0$. This implies $(f_1(x)g_0(x))^2 = 0$ and so (since $R[x]$ is reduced)

(D) $f_1(x)g_0(x) = 0$.

This implies (on account of (B)) that

(E) $f_0(x)g_1(x) = 0$.

Now (A), (D) and (E) yield (since R is Armendariz)

$a_ib_j = 0$, $a_iv_j = 0$ and $u_ib_j = 0$ for each i and j. It follows that

$(a_i, u_i)(b_j, v_j) = (a_ib_j, a_iv_j + u_ib_j) = 0$ for each i and j.

The following generalisation of 2.5 has a similar proof.

2.6. Proposition. Let R be a reduced ring and A an ideal of R such that R/A is reduced. Then R $(+)$ R/A is Armendariz.

2.7. Remark. Recall that a ring R is *strongly regular* ([3], §4) if for each element a in R, there exists an element b in R such that $a = a^2b$. A ring is strongly regular. if and only if it is (von Neumann) regular and reduced. If R is a strongly regular ring, then for each ideal A of R R/A is strongly regular and reduced. On applying 2.6 we get the following result: if R is a strongly regular ring, then for each ideal A of R, the ring R $(+)$ R/A is Armendariz.

We conclude this section with a few more examples of Armendariz rings.

2.8. Proposition. Let K be a field, $h: K \to K$ a field monomorphism, and V a K-vector space. Then the ring K $(+)_h$ V is Armendariz.

Proof. The map h induces a natural ring homomorphism $h: K[x] \to K[x]$. We have the torsion free "polynomial module" $V[x]$ over $K[x]$. We identify $\{K$ $(+)_h$ $V\}[x]$ with $K[x]$

$(+)_h V[x]$. (See 1.3 for definitions).
Now let $f(x), g(x) \in \{K(+)_h V\}[x]$ satisfy $f(x)g(x) = 0$. Write $f(x)$ and $g(x)$ as $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$, where $f_0(x), g_0(x) \in K[x]$ and $f_1(x), g_1(x)$ belong to the polynomial module $V[x]$.
Then $f(x)g(x) = 0 \Rightarrow (f_0(x), f_1(x))(g_0(x), g_1(x)) = 0$
$\Rightarrow (f_0(x)g_0(x), h(f_0(x))g_i(x) + g_0(x)f_i(x)) = 0$
$$\Rightarrow \begin{cases} f_0(x)g_0(x) = 0 \text{ and} \\ h(f_0(x))g_1(x) + g_0(x)f_1(x) = 0. \end{cases}$$

Since the cases $f(x) = 0$ or $g(x) = 0$ are trivial, we look at other cases.

Case 1. $f_0(x) = 0$ but $f_1(x) \neq 0$. Then $h(f_0(x)) = 0 \Rightarrow g_0(x)f_0(x) = 0$ which gives $g_0(x) = 0$ since $V[x]$ is $K[x]$-torsion free.

Case 2. $g_0(x) = 0$ but $g_1(x) \neq 0$. Then $h(f_0(x))g_1(x) = 0$. This implies that $h(f_0(x)) = 0$ by an argument similar to that in Case 1. Since h is a one-one map it follows that $f_0(x) = 0$. Therefore in either of the cases $f(x), g(x)$ must be of the types $f(x) = (0, f_1(x))$, $g(x) = (0, g_1(x))$. If follows that $K(+)_h V$ is Armendariz.

2.9. Corollary. If K is a field and V a K-vector space, then $K(+)V$ is a commutative Armendariz ring which is not reduced if $V \neq 0$.

Proof. Let h be the identity map in Proposition 2.8.

3. Rings which do not have the Armendariz property. In this section we shall give a few examples of rings which are not Armendariz.

3.1. Remark. Full matrix rings of degree ≥ 2 over any ring with identify are non-Armendariz. Consider the polynomials $f(x) = E_{12}x + E_{11}$, $g(x) = E_{11}x - E_{21}$. Then $f(x)g(x) = 0$ but $E_{11}E_{11} = E_{11} \neq 0$.

3.2. Example. Commutative rings need not be Armendariz. Consider the polynomial $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x$ over the ring $\{\boldsymbol{Z}/8\boldsymbol{Z}(+)\boldsymbol{Z}/8\boldsymbol{Z}\}$. The square of this polynomial is zero but the product $(\bar{4}, \bar{0})(\bar{4}, \bar{1}) = (\bar{0}, \bar{4})$ is not zero.

3.3. Remark. The ring considered in 3.2 is a factor ring of an Armendariz ring, namely the ring of polynomials in many variables over $\boldsymbol{Z}$. It is also a factor ring of $\boldsymbol{Z}(+)\boldsymbol{Z}/8\boldsymbol{Z}$ which is Armendariz by 2.4. Thus factor rings of Armendariz rings need not be Armendariz.

4. Other classes of rings. In this section we shall record a few results which connect Armendariz rings to some other classes of rings. We introduce the following definition.

4.1. Definition. A ring R is *a left McCoy ring* if whenever $g(x)$ is a right zero-divisro in $R[x]$ there exists a non-zero element c in R such that $cg(x) = 0$. Right McCoy rings are defined dually. A ring is *a McCoy ring* if it is both left as well as right McCoy.

4.2. Remark. It was proved by McCoy [5] that commutative rings have the above property; for an inductive proof of this result see [7]; see also [2]. If T is a ring with identity, the matrix ring $M_2(T)$ is neither left nor right McCoy. (There do not exist nonzero matrices C, D satisfying $Cg(x) = 0$ and $f(x)D = 0$ for the polynomials considered in Remark 3.1.)

4.3. Remark. Let R be an Armendariz ring and assume that $g(x)$ is a right zero-divisor in $R[x]$. Then there exists a non-zero polynomial $f(x) \in R[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz, $a_ib_j = 0$ for each i and j. Since $f(x) \neq 0$, $a_t \neq 0$ for some t; clearly $a_tg(x) = 0$. Thus R is left (similarly right) McCoy. This shows that Armendariz rings are McCoy. The converse is not true; commutative rings are McCoy, as noted in 4.2, but we have examples of commutative non-Armendariz rings.

4.4. Definition ([3], §4). A ring R is *semi-commutative* if it satisfies the following condition: whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R.

4.5. Remarks and questions. The class of commutative rings and the class of reduced rings are contained in the class of semi-commutative rings. Both these (smaller) classes are trivially stable under the formation of polynomial rings.

A ring R is called *normal* if every idempotent in R is central; semi-commutative rings are normal ([3], Lemma 5). Against this background consider the following "stability" assertions:
(i) R normal $\Rightarrow R[x]$ normal;
(ii) R semi-commutative $\Rightarrow R[x]$ semi-commutative; and
(iii) R Armendariz $\Rightarrow R[x]$ Armendariz,

We remark that (i) easily follows from an extension of ([1], Corollary 1) to normal rings. (It may be a known result but we have not seen a proof of (i) in the literature).

We do not know whether (ii) and (iii) are true. In view of these questions, the following proposition may be of some interest.

4.6. Proposition. If R is a semi-commutative ring which is Armendariz, then $R[x]$ is semi-commutative.

Proof. Let $f(x)$, $g(x)$ be polynomials in $R[x]$ satisfying $f(x)g(x) = 0$. Let $h(x) = \sum_{k=0}^{t} c_k x^k \in R[x]$. Since R is Armenderiz and $f(x)\, g(x) = 0$, $a_i b_j = 0$ for each i and j. Since R is semi-commutative $a_i c_k b_j = 0$ for each i, j and k. Hence $f(x)h(x)g(x) = 0$. This proves that $R[x]$ is semi-commutative.

4.7. Remark. The concepts introduced and studied in this note have extensions in the context of modules, graded rings and graded modules. Related concepts can also be defined for power series rings. These generalisations will be carried out elsewhere.

Acknowledgement. We thank the referee for a careful reading of the manuscript, which led to improvements in presentation.

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