# Identification of the Potential Term of the Wave Equation

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**Abstract:** We consider inverse problems of recovering coefficients of the wave equation from the Cauchy data and the boundary data on a part of lateral surface of a non-convex domain. We show uniqueness theorems for the inverse problems of determination of the source term or the potential one of the wave equation, and our results are solutions to an open problem.

**1. Introduction.** The main interest of our research lies in the problem to identify coefficients of hyperbolic equations from lateral boundary data. In geophysics, it is one of the main problems to determine properties of a physical medium from observations on boundaries of a domain, and our research is a mathematical aspect of the problem.

In this paper we present uniqueness theorems of inverse problems for the wave equation. Inverse problems to identify the source term or the potential one have been dealt with by many researchers; e.g. Bukhgeĭm [1], Bukhgeĭm and Klibanov [2], Isakov [6], Khaĭdarov [7], Klibanov [8]. They pose some technical hypotheses for the uniqueness, and the uniqueness of identification in generic cases has been open. The main results of our research are solutions to the open problem.

The key ideas are an application of the Carleman estimates due to Bukhgeĭm and Klibanov [2] and an application of the Gauss-Fourier transformation introduced by Robbiano [10]. The method proposed in [2] is effective for inverse problems to identify coefficients of any type of equation for which the Carleman estimates hold, but we cannot expect that the ordinary Carleman estimate holds for our problem. We use the idea of Robbiano [10] to apply the Fourier-Gauss transformation to change the problem into a form which Bukhgeĭm's technique can be applied to.

**2. Notations and results.** Let $G$ be a half space such that

$$G := \{x \in \boldsymbol{R}^n : x_n > 0\},$$

and define an operator $P$ by

$$P := \partial_t^2 - \Delta_x - q(x) \quad (x, t) \in G \times \boldsymbol{R}_t^1, \tag{2.1}$$

where

$$q(x) \in C^0(\bar{G}).$$

We firstly consider the inverse problem to identify the source term from the measurement of the solution on a lateral boundary at $-T \leq t \leq T$, where $T$ is a positive constant given a priori. Let $w(x, t)$ be a given function in $C^3(\bar{G} \times [-T, T])$, and consider an initial-boundary value problem for the wave equation:

$$Pu(x, t) = \rho(x) w(x, t) \quad (x, t) \in G \times (-T, T), \tag{2.2}$$

$$u(x, 0) = f^0(x),\ \partial_t u(x, 0) = f^1(x) \quad x \in G, \tag{2.3}$$

$$u(x, t) = g^0(x, t) \quad (x, t) \in \partial G \times (-T, T). \tag{2.4}$$

Furthermore we observe the lateral Neumann boundary data

$$\frac{\partial u}{\partial \nu}(x, t) = g^1(x, t) \quad (x, t) \in \partial G \times (-T, T), \tag{2.5}$$

and our problem is to seek a pair of unknown functions $\{\rho(x), u(x, t)\}$ which satisfy (2.2)-(2.5). Here $f^0(x), f^1(x), g^0(x, t)$ and $g^1(x, t)$ are given.

We introduce the following notations:

$$V_0^K := \{(x, t) \in G \times (-T, T) : |t| < T - K|x|\}, \tag{2.6}$$

$$G_0^K := V_0^K \cap \{t = 0\},\ \Gamma_0^K := \partial G_0^K \cap \partial G, \tag{2.7}$$

where $K = \sqrt{27/23}$. The constant $K = 1$ might be natural for our problem, but unfortunately it seems impossible. (See Hörmander [5]).

**Theorem 2.1** (Main result). *Suppose*

$$\rho(x) \in C(\overline{G_0^K}), \quad w(x, t), \quad u(x, t) \in C^3(\overline{G_0^K} \times [-T, T]),$$

*and assume* $w(x, 0) \neq 0$ $(x \in \overline{G_0^K})$.

*If a pair of functions* $\{\rho(x), u(x, t)\}$ *satisfy*

$$Pu(x, t) = \rho(x) w(x, t) \quad (x, t) \in G_0^K \times (-T, T),$$

$$u(x, 0) = 0, \quad \partial_t u(x, 0) = 0 \quad x \in G_0^K,$$

$$u(x, t) = 0 \quad (x, t) \in \Gamma_0^K \times (-T, T),$$

$$\frac{\partial u}{\partial \nu}(x, t) = 0 \quad (x, t) \in \Gamma_0^K \times (-T, T),$$

*then we have*

$$\rho(x) = 0 \quad x \in G_0^K,$$
$$u(x, t) = 0 \quad (x, t) \in V_0^K.$$

We secondary consider the inverse problem to recover the potential term in the operator $P$: We identify the potential term $q(x)$ in $P$ for a known source term, and our aimed solution is a pair of functions $\{q(x), u(x, t)\}$. We show the following uniqueness theorem for the inverse problem, and it follows immediately from Theorem 2.1.

**Theorem 2.2.** *Suppose*

$$q_j \in C(\overline{G_0^K}), \quad u_j \in C^3(\overline{G_0^K} \times [-T, T]) \ j = 1,2.$$

*If the each pair of the functions* $\{q_j(x), u_j(x, t)\}_{j=1,2}$ *satisfy*

$$(\partial_t^2 - \Delta_x - q_j(x))u_j(x, t) = F(x, t) \quad (x, t) \in G_0^K \times (-T, T),$$
$$u_j(x, t) = f^0(x), \quad \partial_t u_j(x, 0) = f^1(x) \quad x \in G_0^K,$$
$$u_j(x, t) = g^0(x, t) \quad (x, t) \in \Gamma_0^K \times (-T, T),$$
$$\frac{\partial u_j}{\partial \nu}(x, t) = g^1(x, t) \quad (x, t) \in \Gamma_0^K \times (-T, T),$$

*and if* $f^0(x) \neq 0$ $(x \in \overline{G_0^K})$, *then we have*

$$q_1(x) = q_2(x) \quad x \in G_0^K,$$
$$u_1(x, t) = u_2(x, t) \quad (x, t) \in V_0^K,$$

*Proof.* Set

$$\rho(x) := q_1(x) - q_2(x), \ w(x, t) := u_2(x, t),$$
$$u(x, t) := u_1(x, t) - u_2(x, t),$$

then the second inverse problem is reduced to the first one, and $\{\rho(x), u(x, t)\}$ satisfies the hypotheses of the Theorem 2.1. Hence the proof is complete. □

**3. Outline of Proof Theorem 2.1.** The complete proof of our results will be given in Kubo [9], and we only show a sketch of the proof of Theorem 2.1. We change our hyperbolic problem into some kind of elliptic problem through the localized Fourier-Gauss transformation due to Robbiano [10], and we apply Bukhgeĭm's technique which is based on the Carleman estimates. The following lemmas are key estimates in our proof.

Let us define an operator $Q$ by

(3.1) $Q := -\partial_s^2 - \Delta_x - q(x) \quad (x, s) \in G \times \boldsymbol{R}_s^1.$

The operator $Q$ corresponds to a transformed form of $P$, and we consider the Carleman estimate for the operator $Q$. Let $\varphi(x, s)$ denote a weight function

(3.2) $\varphi(x, s) := \exp\{\psi(x, s)\} - 1,$

where

$$\psi(x, s) := 1 - \left(\frac{s^2}{S^2} + \frac{|x + \sigma|^2}{S^2}\right),$$
$$\sigma := (0,0,\ldots, 0, \sigma_n).$$

Let us define a domain $\Omega_\varepsilon$ by

$$\Omega_\varepsilon := \{(x, s) \in G \times \boldsymbol{R}_s^1 : \varphi(x, s) > \varepsilon\},$$

and let us denote the weighted norm for the Carleman estimate by

$$\|u\|_{\gamma,m}^2 = \|u\|_{\gamma,m,\Omega_0}^2 := \sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)} \|\exp(\gamma\varphi)D^\alpha u\|_{L^2(\Omega_0)}^2.$$

Then we obtain the following Carleman estimate. For details, see Hörmander [3] and [4].

**Lemma 3.1.** *For positive constants* $\sigma_n$ *and* $S$, *there exist positive constants* $\gamma_0$ *and* $C$ *such that for any* $\gamma > \gamma_0$

(3.3) $\|u\|_{\gamma,2}^2 \leq C\gamma \|Qu\|_{\gamma,0}^2, \quad \forall u \in C_0^\infty(\Omega_0).$

For $\lambda > 0$ let us define an integral transformation for $u(x, t) \in C(G_0^K \times [-T, T])$;

(3.4) $(\Lambda_{a,\lambda}u)(x, s) := \sqrt{\dfrac{\lambda}{2\pi}} \displaystyle\int_{-T}^{T} e^{-\frac{\lambda}{2}(is+a-t)^2} u(x, t)\,dt.$

We call it a localized Fourier-Gauss transformation, which satisfies the following properties.

**Lemma 3.2.** *Let* $u \in C^2(G_0^K \times [-T, T])$. *Then there exists a positive number* $C$ *such that for any* $\lambda > 0$

(3.5) $$\|(Q\Lambda_{\alpha,\lambda} - \Lambda_{\alpha,\lambda}P)u\|_{H^1(\Omega_0)} \leq C\lambda^{\frac{5}{2}} \exp\left\{-\frac{\lambda}{2}[(T - |a|)^2 - S^2]\right\}.$$

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