Duality for Hypergeometric Period Matrices

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We present some basic identities for the hypergeometric period matrices associated with the integrals of Euler type. Our main theorem shows not only identities classically known for integrals expressing hypergeometric series such as

$$(1)\qquad \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^a(1-t)^{c-a}(1-tx)^{-b}(1-ty)^{-b'}\frac{dt}{t(1-t)}$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\iint_{\substack{s>0,t>0\\1-s-t>0}} s^b t^{b'}(1-s-t)^{c-b-b'}(1-sx-ty)^{-a}\frac{ds\wedge dt}{st(1-s-t)}$$

but also identities for various hypergeometric functions. The full context of the theory will be published elsewhere.

Let $M(k+1,n+2)$ be the set of $(k+1)\times(n+2)$ complex matrices such that any $(k+1)$-minor does not vanish; for an element $x=(x_{ij})_{0\le i\le k, 0\le j\le n+1}\in M(k+1,n+2)$, put

$$L_j=L_j(t,x)=\sum_{i=0}^{k} t_i x_{ij},$$
$$H_j=H_j(x)=\{t\in \boldsymbol{P}^k \mid L_j(t,x)=0\},$$
$$T(x)=\boldsymbol{P}^k-\bigcup_{j=0}^{n+1} H_j(x),$$
$$x\langle J\rangle=\det(x_{ij_m})_{0\le i,m\le k},$$

where $t=(t_0,\dots,t_k)$ is a homogeneous coordinate system of the complex projective space $\boldsymbol{P}^k$ and $J=\{j_0,\cdots,j_k\}$, $0\le j_0<j_1<\cdots<j_k\le n+1$ denotes a multi-index. We define a multi-valued function $U^\alpha=U^\alpha(t,x)$ and holomorphic k-forms $\varphi_J=\varphi_J(t,x)$ on $T(x)\times M(k+1,n+2)$ by

$$U^\alpha(t,x)=\prod_{j=0}^{n+1} L(t,x)^{\alpha_j}/\prod_J x\langle J\rangle^{(\alpha_{j_0}+\cdots+\alpha_{j_k})/\binom{n}{k}},$$
$$\varphi_J(t,x)=d_t\log(L_{j_0}(t,x)/L_{j_1}(t,x))\wedge\cdots\wedge d_t\log(L_{j_{k-1}}(t,x)/L_{j_k}(t,x)),$$

where

$$\alpha=(\alpha_0,\cdots,\alpha_{n+1}),\ \alpha_j\in \boldsymbol{C}\setminus\boldsymbol{Z},\ \textstyle\sum_{j=0}^{n+1}\alpha_j=0.$$

Let ξ_k be a fixed element of $M(k+1,n+2)$ of the following form:

$$\xi_k=\begin{pmatrix}1&1&\cdots&1&0\\ \lambda_0&\lambda_1&\cdots&\lambda_n&0\\ \lambda_0^2&\lambda_1^2&\cdots&\lambda_n^2&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \lambda_0^k&\lambda_1^k&\cdots&\lambda_n^k&1\end{pmatrix},$$
$$0\le\lambda_0<\lambda_1<\cdots<\lambda_n;$$

since $\xi_k\langle J\rangle$ is positive for any J, we assign the argument of $\xi_k\langle J\rangle$ by requiring $\arg(\xi_k\langle J\rangle)=0$. Let $\Delta_J=\Delta_J(\xi_k)$ be the simplex in $\boldsymbol{P}^k(\boldsymbol{R})\subset\boldsymbol{P}^k$ defined by the inequalities

$$(-1)^{k-m}(L_{j_m}(t,\xi_k)/L_{n+1}(t,\xi_k))>0,\quad j_m\in J;$$

we assign the argument of L_{j_m}/L_{n+1} on Δ_J by

$$\arg(L_{j_m}(t,\xi_k)/L_{n+1}(t,\xi_k))=(k-m)\pi.$$

Note that $\Delta_J\cap H_j\ne\phi$ for $j_m<j<j_{m+1}$; we deform Δ_J to $\Delta_J^+=\Delta_J^+(\xi_k)\subset T(\xi_k)$ so that it is avoiding H_j, $j_m<j<j_{m+1}$ and that the arguments of L_j/L_{n+1} are assigned by

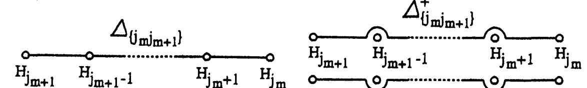
$$(k-m-1)\pi\le\arg(L_j(t,\xi_k)/L_{n+1}(t,\xi_k))\le(k-m)\pi,\ \text{for } j_m<j<j_{m+1}.$$

Let $\Delta_J^-=\Delta_J^-(\xi_k)$ be a deformation of Δ_J near H_j, $j\notin J$ on which the arguments of L_j/L_{n+1} are assigned by

$$\arg(L_{j_m}(t,\xi_k)/L_{n+1}(t,\xi_k))\doteqdot-(k-m)\pi,\quad \text{for } j_m\in J$$
$$-(k-m)\pi\le\arg(L_j(t,\xi_k)/L_{n+1}(t,\xi_k))\le-(k-m-1)\pi,\ \text{for } j_m<j<j_{m+1};$$

see the following figure.

$k=1$



$k=2$

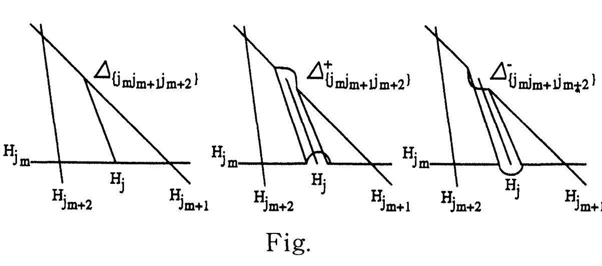


Fig.

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The assignment of arguments above defines the branch $U^{\alpha}_{\Delta^{\pm}_J}(t, \xi_k)$ of U^{α} on $\Delta^{\pm}_J$. Let $\gamma^{+}_J(\alpha ; t, \xi_k)$ denote the pair of the simplex Δ^{+}_J and the branch $U^{\alpha}_{\Delta^{+}_J}(t, \xi_k)$ and $\gamma^{-}_J(\alpha ; t, \xi_k)$ the pair of Δ^{-}_J and $U^{\alpha}_{\Delta^{-}_J}(t, \xi_k)$. For general $x \in M(k+1, n+2)$, we continuously deform $\gamma^{+}_J(\alpha ; t, \xi_k)$ and $\gamma^{-}_J(\alpha ; t, \xi_k)$ along a path ρ from ξ_k to x in $M(k+1, n+2)$, thus defining $\gamma^{+}_J(\alpha) = \gamma^{+}_J(\alpha ; t, x)$ and $\gamma^{-}_J(\alpha) = \gamma^{-}_J(\alpha ; t, x)$; note that they depend on the choice of ρ.

We define functions $F^{+}_{IJ}(\alpha ; x)$ and $F^{-}_{IJ}(\alpha ; x)$ on $M(k+1, n+2)$ by

$$F^{\pm}_{IJ}(\alpha ; x) = \langle \varphi_1(t, x), \gamma^{\pm}_J(\alpha ; t, x)\rangle = \int_{\Delta^{\pm}_J(x)} U^{\alpha}_{\Delta^{\pm}_J(x)}(t, x)\varphi_I(t, x),$$

where I and J are multi-indices; we do not need to worry about divergence of the integrals (cf. [7]). Since $\gamma^{\pm}_J(\alpha ; t, x)$ depend on the choice of a path ρ from ξ_k to x, $F^{\pm}_{IJ}(\alpha ; x)$ are multi-valued on $M(k+1, n+2)$; refer to [8] for monodromy groups.

For $g \in GL_{k+1}(\boldsymbol{C})$ and $r = (r_0, \cdots, r_{n+1}) \in (\boldsymbol{C}^*)^{n+2}$, take a path $g(s)$ from 1_{k+1} to g in $GL_{k+1}(\boldsymbol{C})$ and a path $r(s)$ from $(1, \cdots, 1)$ to r in $(\boldsymbol{C}^*)^{n+2}$ in order to define $F^{\pm}_{IJ}(\alpha ; g\cdot x\cdot r)$ by the continuation of $F^{\pm}_{IJ}(\alpha ; x)$ along the path $gxr(s) = g(s)\cdot x\cdot \mathrm{diag}(r_0(s), \cdots, r_{n+1}(s))$ in $M(k+1, n+2)$. The facts that $gxr(s)\langle J\rangle = \det(g(s))\, x\langle J\rangle \prod_{j\in J} r_j(s)$, $L_j(t, gxr(s)) = L_j(tg(s), x) r_j(s)$, $\sum_{j=0}^{n+1}\alpha_j = 0$, and that the integrals are independent of the choice of the coordinate t, show

$$F^{\pm}_{IJ}(\alpha ; g\cdot x\cdot r) = F^{\pm}_{IJ}(\alpha ; x)$$

for any choice of the paths.

Lemma. *The functions $F^{\pm}_{IJ}(\alpha ; x)$ are defined on the double quotient space*

$$X(k+1, n+2) = GL_{k+1}(\boldsymbol{C}) \backslash M(k+1, n+2)/(\boldsymbol{C}^*)^{n+2}.$$

We call $X(k+1, n+2)$ the configuration space of $n+2$ hyperplanes on $\boldsymbol{P}^k$ in general position and denote by $[x]$ the element of $X(k+1, n+2)$ represented by $x \in M(k+1, n+2)$.

Let I_0, I_{n+1} and J_0, J_{n+1} be multi-indices of type

$$I_0 = \{0, i_1, \cdots, i_k\},\ I_{n+1} = \{i_1, \cdots, i_k, n+1\},\quad 1 \le i_1 < \cdots < i_k \le n,$$
$$J_0 = \{0, i_1, \cdots, i_k\},\ J_{n+1} = \{j_1, \cdots, j_k, n+1\},\quad 1 \le j_1 < \cdots < j_k \le n.$$

Definition. *The $\binom{n}{k} \times \binom{n}{k}$ matrices*

$$\Pi^{+}_0(\alpha ; [x]) = (F^{+}_{I_0J_0}(\alpha ; x))_{I_0J_0} \text{ and}$$
$$\Pi^{-}_{n+1}(\alpha ; [x]) = (F^{-}_{I_{n+1}J_{n+1}}(\alpha ; x))_{I_{n+1},J_{n+1}},$$

where I_0's, J_0's and I_{n+1}'s, J_{n+1}'s are arranged lexicographically, are called the hypergeometric period matrices of type (k, n) with parameter α on the configuration space $X(k+1, n+2)$.

The spaces $X(k+1, n+2)$ and $X(l+1, n+2)$, $l = n-k$, are isomorphic; an isomorphism $\perp$ is given as follows: for $x \in M(k+1, n+2)$, there uniquely exists $x^{\perp} \in M(l+1, n+2)$ modulo $SL_{l+1}(\boldsymbol{C})$ such that $x\langle J\rangle = x^{\perp}\langle J^{\perp}\rangle$, where $J = \{j_0, j_1, \ldots, j_k\}$, $J^{\perp} = \{j_{k+1}, \cdots, j_n, j_{n+1}\}$, $0 \le j_{k+1} < \cdots < j_{n+1} \le n+1$, $J \cup J^{\perp} = \{0,1, \cdots, n, n+1\}$. The isomorphism $\perp$ is defined by

$$\perp : X(k+1, n+2) \ni [x] \mapsto [x]^{\perp} = [x^{\perp}] \in X(l+1, n+2).$$

For $g \in GL_n(\boldsymbol{C})$, put

$$\wedge^k g = (\det(g_{pq})_{p\in P, q\in Q})_{PQ} \in GL_{\binom{n}{k}}(\boldsymbol{C}),$$
$$P = \{p_1, \cdots, p_k\},\ Q = \{q_1, \cdots, q_k\},$$

where P's and Q's are arranged lexicographically; we have

$$\wedge^k(g_1g_2) = (\wedge^k g_1)(\wedge^k g_2),\ g_1, g_2 \in GL_n(\boldsymbol{C}).$$

Main Theorem (*Duality for hypergeometric period matrices*).

$$\Pi^{+}_0(\alpha ; [x]) = V(\alpha)\, E_{kn}(\wedge^l I_{ch}(\alpha)^{-1})\, \Pi^{-}_{n+1}(-\alpha ; [x]^{\perp})(\wedge^l I_h(\alpha)^{-1})^t E_{kn} \tag{2}$$

where

$$V(\alpha) = e^{n\pi\sqrt{-1}\alpha_0} e^{(n-1)\pi\sqrt{-1}\alpha_1} \cdots e^{\pi\sqrt{-1}\alpha_{n-1}} \frac{\Gamma(\alpha_0)\cdots\Gamma(\alpha_n)}{\Gamma(-\alpha_{n+1})},$$

$$I_{ch}(\alpha) = -2\pi\sqrt{-1}\,\mathrm{diag}(1/\alpha_1, 1/\alpha_2, \cdots, 1/\alpha_n),$$

$$I_h(\alpha) = \mathrm{diag}\Big(\frac{e^{2\pi\sqrt{-1}(\alpha_0+\alpha_1)}}{e^{2\pi\sqrt{-1}\alpha_1}-1}, \frac{e^{2\pi\sqrt{-1}(\alpha_0+\alpha_1+\alpha_2)}}{e^{2\pi\sqrt{-1}\alpha_2}-1}, \cdots, \frac{e^{2\pi\sqrt{-1}(\alpha_0+\cdots+\alpha_n)}}{e^{2\pi\sqrt{-1}\alpha_n}-1}\Big),$$

and $E_{kn} = (e_{I_0J_0^{\perp}})_{I_0J_0^{\perp}}$ is an anti-diagonal $\binom{n}{k} \times \binom{n}{k}$ matrix given by

$$e_{I_0J_0^{\perp}} = \begin{cases} (-1)^{j_1+\cdots+j_k} & \text{if } I_0 = J_0 = \{0, j_1, \cdots, j_k\}, \\ 0 & \text{if } I_0 \ne J_0. \end{cases}$$

The path from $[\xi_l] = [\xi_k]^{\perp}$ to $[x]^{\perp}$ defining $\Pi^{-}_{n+1}(-\alpha ; [x]^{\perp})$ is the $\perp$-image of the path defining $\Pi^{+}_0(\alpha ; [x])$.

Remark 1. The duality says componentwise

$$F^{+}_{I_0J_0}(\alpha ; [x]) = cF^{-}_{I_0^{\perp}J_0^{\perp}}(-\alpha ; [x]^{\perp})$$

for a constant c which can be expressed in terms of components of intersection matrices $I_{ch}(\alpha) =$

$\langle\varphi_{\{0,i\}}, \varphi_{\{j,n+1\}}\rangle_{ij}$ and $I_h(\alpha) = \langle\gamma^+_{\{0,i\}}(\alpha), \gamma^-_{\{j,n+1\}}\alpha(-\alpha)\rangle_{ij}$; these matrices were studied in [2] and [6], respectively.

To prove the main theorem, we need three propositions whose proofs are referred to [1], [2], [5], [6] and [9].

Proposition 1 (*Invariant Gauss-Manin systems*). *The hypergeometric period matrix* $\Pi_0^+(\alpha; [x])$ *satisfies the following differential equation*

$$d\Pi_0^+(\alpha; [x]) = \Theta^\alpha([x])\Pi_0^+(\alpha; [x]),$$

where $\Theta^\alpha([x]) = (\theta^\alpha_{I_0J_0})_{I_0J_0}$ *is given by*

$$\theta^\alpha_{J_0J_0} = \sum_{j_m\in J_0}\alpha_{j_m}\, d\log\frac{x\langle J^{n+1\setminus 0}_0\rangle}{x\langle J_0^{n+1\setminus j_m}\rangle} + \sum_{j\in J_0^\perp}\alpha_j\, d\log\frac{x\langle J_0^{j\setminus 0}\rangle}{x\langle J_0\rangle} - \frac{1}{\binom{n}{k}}\sum_J(\sum_{j\in J}\alpha_j)\, d\log x\langle J\rangle,$$

$$\theta^\alpha_{J_0J_0^{j\setminus j_m}} = (-1)^{m+\#\{i\in J_0\setminus\{j_m\}|i<j\}}\alpha_j\, d\log\frac{x\langle J_0^{j\setminus j_m}\rangle x\langle J_0^{n+1\setminus 0}\rangle}{x\langle J_0^{j\setminus 0}\rangle x\langle J_0^{n+1\setminus j_m}\rangle},$$

$\theta^\alpha_{I_0J_0} = 0$ *for other* (I_0, J_0),

here $J_0^{j\setminus j_m}$ *denotes the multi-index* $(J_0\setminus\{j_m\})\cup\{j\}$, $j_m\in J_0$, $j\in J_0^\perp$.

Proposition 2 (*Twisted Riemann's period relations for* $k=1$).

(3) $\Pi^-_{n+1}(-\alpha; [\xi_1])I_h(\alpha)^{-1}\,{}^t\Pi_0^+(\alpha; [\xi_1]) = I_{ch}(\alpha)$.

Proposition 3 (*Wedge formulae for period matrices*).

$$\wedge^k\Pi_0^+(\alpha; [\xi_1]) = \Pi_0^+(\alpha; [\xi_k]),$$
$$\wedge^k\Pi^-_{n+1}(\alpha; [\xi_1]) = \Pi^-_{n+1}(\alpha; [\xi_k]),$$

in particular,

$$\wedge^n\Pi_0^+(\alpha; [\xi_1]) = \Pi_0^+(\alpha; [\xi_1]) = \int_{\Delta^+_{\{0,\dots,n\}}} U^\alpha_{\Delta^+_{\{0,\dots,n\}}}\varphi_{\{0,\dots,n\}} = V(\alpha)$$

(*A sketch of a proof of the main theorem.*) A straightforward calculation shows that the right hand side of (2) satisfies the differential equation in Proposition 1. Thus we have only to show the identity (2) at the point $[\xi_k]$. By the property $[\xi_l]^\perp = [\xi_k]$ and by taking the l-fold wedge product of (3), we have

$$(\wedge^l I_{ch}(\alpha)^{-1})(\wedge^l\Pi^-_{n+1}(-\alpha; [\xi_1]))(\wedge^l I_h(\alpha)^{-1}) = {}^t(\wedge^l\Pi_0^+(\alpha; [\xi_1]))^{-1};$$

then Laplace's expansion formula and Proposition 3 conclude the main theorem.

Remark 2. When $k=1$ and $n=3$, we can see the equality in (1) by comparing the both sides of the top-left components of (2) for parameters

$$\alpha_0 = a,\ \alpha_1 = c-a\ \ \alpha_2 = -b,$$
$$\alpha_3 = -b',\ \alpha_4 = b+b'-c,$$

and by using the formula

$$\Gamma(a)\Gamma(-a) = \frac{2\pi\sqrt{-1}}{a}\frac{-e^{\pi\sqrt{-1}a}}{e^{2\pi\sqrt{-1}a}-1}.$$

The top-row vectors of (2) lead the identities among single integrals and double integrals that are fundamental solutions of Appell's F_1. For general (k, n), the both sides of the top-left components of (2) are expressed by the hypergeometric series of type $(k+1, n+2)$ studied in [7], and the I_0-th row vectors of (2) lead the identities among k-fold integrals and l-fold integrals that are fundamental solutions of the hypergeometric differential equation of type $(k+1, n+2)$ with parameters $\alpha(I_0) = (\cdots, \alpha_i(I_0), \cdots)$, where

$$\alpha_i(I_0) = \begin{cases}\alpha_i - 1 & \text{if } i\in I_0,\\ \alpha_i & \text{if } i\notin I_0.\end{cases}$$

Refer to [3], [4] and [7].

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