26. *On a Conjecture of Shanks*

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§1. Introduction. The purpose of the present article is to give some refinements of the previous works [2]-[6] concerning Shanks' conjecture.

Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In explaining theoretically a strange tendency which appears when one draws the graph of $\zeta\left(\frac{1}{2} + it\right)$ for $t \geq 0$ in the complex plain, Shanks [8] has given the following conjecture.

Conjecture. $\zeta'\left(\frac{1}{2} + i\gamma\right)$ *is positive real in the mean.*

Concerning this, we can show the following theorems. We suppose always that $T > T_0$ and C denotes some positive constant. Let R. H. be the abbreviation of the Riemann Hypothesis. Let C_0 and C_1 be the Laurent coefficients in

$$\zeta(s) = \frac{1}{s-1} + C_0 + C_1(s-1) + \dots .$$

Theorem 1. $$\sum_{0<\gamma\leq T} \zeta'(\rho) = \frac{1}{4\pi} T \log^2 \frac{T}{2\pi} + (C_0 - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (C_1 - C_0)\frac{T}{2\pi} + O(T\exp(-C\sqrt{\log T})).$$

Theorem 2 (Under R. H.). $$\sum_{0<\gamma\leq T} \zeta'\left(\frac{1}{2} + i\gamma\right) = \frac{1}{4\pi} T \log^2 \frac{T}{2\pi} + (C_0 - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (C_1 - C_0)\frac{T}{2\pi} + O(T^{\frac{1}{2}}\log^{\frac{7}{2}} T).$$

These imply that $\zeta'(\rho)$ is positive real in the mean and also improve upon both our previous results [2][4][6] and also Conrey-Gohsh-Gonek [1]. Theorem 1 is announced in [6].

On the other hand, the following two theorems may provide us an explanation of the strange tendency mentioned above.

Theorem 3. *For* $0 \neq \Delta = 2\pi\alpha/\log(T/2\pi) \ll 1$, *we have*

$$\sum_{0<\gamma\leq T} \zeta(\rho + i\Delta) = \pi\alpha\left(\frac{1 - \frac{\sin 2\pi\alpha}{2\pi\alpha}}{\pi\alpha} + i\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+ \frac{T}{2\pi}\left(-1 + \left(\frac{T}{2\pi}\right)^{-i\Delta}\frac{1}{i\Delta}\left(\frac{1}{1-i\Delta} - 1\right) - \left(\frac{T}{2\pi}\right)^{-i\Delta}\frac{1}{1-i\Delta}\left(\zeta(1-i\Delta) + \frac{1}{i\Delta}\right)\right.$$
$$\left. + \left(\frac{\zeta'}{\zeta}(1+i\Delta) + \frac{1}{i\Delta}\right)\right) + O(T\exp(-C\sqrt{\log T})).$$

Theorem 4 (Under R. H.). *For* $0 \neq \Delta = 2\pi\alpha/\log(T/2\pi) \ll 1$, *we have*

$$\sum_{0<\gamma\le T} \zeta\Big(\frac{1}{2} + i\Big(\gamma + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\Big)\Big) = \pi\alpha\Big(\frac{1-\frac{\sin 2\pi\alpha}{2\pi\alpha}}{\pi\alpha} + i\Big(\frac{\sin\pi\alpha}{\pi\alpha}\Big)^2\Big)\frac{T}{2\pi}\log\frac{T}{2\pi}$$

$$+\frac{T}{2\pi}\Big(-1+\Big(\frac{T}{2\pi}\Big)^{-i\Delta}\frac{1}{i\Delta}\Big(\frac{1}{1-i\Delta}-1\Big)-\Big(\frac{T}{2\pi}\Big)^{-i\Delta}\frac{1}{1-i\Delta}\Big(\zeta(1-i\Delta)+\frac{1}{i\Delta}\Big)$$

$$+\Big(\frac{\zeta'}{\zeta}(1+i\Delta)+\frac{1}{i\Delta}\Big)\Big)+O(T^{\frac{1}{2}}\log^{\frac{5}{2}} T).$$

These improve upon our previous results in [5][6]. The graph of

$$\pi\alpha\Big(\frac{1-\frac{\sin 2\pi\alpha}{2\pi\alpha}}{\pi\alpha} + i\Big(\frac{\sin\pi\alpha}{\pi\alpha}\Big)^2\Big)$$

in the complex plane fits very well with the statistical tendency of the graph of p. 85 of Shanks [8].

Here we may mention the method in a few words. In our previous proof we have used the following approximate functional equation as a starting point; for $0 \le \sigma = \Re s \le 1$ and for $t = \Im s \ge t_0$,

$$\zeta(s) = \sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{1}{n^s} + \chi(s)\sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1-s}} + O(t^{-\frac{\sigma}{2}}),$$

where we put $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\Big/\Gamma\Big(\frac{s}{2}\Big)$ with the Γ-function $\Gamma(s)$. Then using our previous results on the sums of the form

$$\sum_{0<\gamma\le T} e^{i\alpha\gamma} \text{ and } \sum_{0<\gamma\le T} e^{i\gamma\log\frac{\gamma}{2\pi e\alpha}},$$

we get our main terms with the worse remainder term. We see from the beginning that by this method the remainder term cannot be better than $O(T^{3/4}\log T)$. Here we shall evaluate directly the complex integral as has been done in Gonek [7], where he has proved among others that under R. H. and for $|\alpha| \le (1/4\pi)\log(T/2\pi)$,

$$\sum_{0<\gamma\le T}\Big|\zeta\Big(\frac{1}{2}+i\Big(\gamma+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\Big)\Big)\Big|^2 = \Big(1-\Big(\frac{\sin\pi\alpha}{\pi\alpha}\Big)^2\Big)\frac{T}{2\pi}\log^2 T + O(T\log T).$$

We have given a refinement of this result in [3]. A further refinement can be obtained by the present method.

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§2. Proof of Theorem 3. We put $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$, and $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$. The functional equation of $\zeta(s)$ is $\zeta(1-s) = \chi(1-s)\zeta(s)$. We may suppose that $T > T_0$ and T is not the imaginary part of the zeros of $\zeta(s)$ and further that $|T-\gamma| \gg \frac{1}{\log T}$ for any γ. This restricton on T is harmless within the remainder term $O(T^{\frac{1}{2}+\varepsilon})$ for any positive ε. We put further $a = 1 + \delta$ with $\delta = 1/\log T$.

We now consider the following integral I around the rectangle R joining the points $a + iC$, $a + iT$, $1 - a + iT$ and $1 - a + iC$, where we suppose that C is a constant satisfying $|\Delta| < C$.

$$I = \frac{1}{2\pi i}\int_R \frac{\xi'}{\xi}(s)\zeta(s + i\Delta)ds.$$

Obviously,

$$I = \sum_{0<\gamma\le T} \zeta(\rho + i\Delta).$$

On the other hand, we have

$$I = \frac{1}{2\pi i}\left(\int_{a+iC}^{a+iT} + \int_{1-a+iT}^{1-a+iC} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iC}^{a+iC}\right)\frac{\xi'}{\xi}(s)\zeta(s + i\Delta)ds$$
$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

Since $\frac{\xi'}{\xi}(s) \ll \log^2 T$ for $-1 \le \sigma \le 2$, we have by Lemma 1 below,

$$I_3 + I_4 \ll \log^2 T\int_{1-a}^{a} |\zeta(\sigma + i(T + \Delta))|\, d\sigma \ll \sqrt{T}\log^2 T.$$

Lemma 1. $\displaystyle\int_{1-a}^{a} |\zeta(\sigma + iT)|\, d\sigma \ll \sqrt{T}.$

Proof. The left hand side is

$$\ll \int_{\frac{1}{2}}^{1}\left(|\zeta(\sigma + iT)| + \left|\frac{\zeta(\sigma - iT)}{\chi(\sigma - iT)}\right|\right)d\sigma + \int_{1}^{a}\left(|\zeta(\sigma + iT)| + \left|\frac{\zeta(\sigma - iT)}{\chi(\sigma - iT)}\right|\right)d\sigma$$
$$\ll \int_{\frac{1}{2}}^{1} T^{\sigma-\frac{1}{2}}|\zeta(\sigma + iT)|\, d\sigma + \int_{1}^{a} T^{\sigma-\frac{1}{2}}|\zeta(\sigma + iT)|\, d\sigma = S_1 + S_2, \text{ say,}$$

where we have used the following property;

$$\chi(\sigma - iT) = e^{-\frac{\pi i}{4}}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{iT\log\frac{T}{2\pi e}}\left(1 + O\left(\frac{1}{T}\right)\right).$$

Using the approximate functional equation for $\zeta(s)$ described above, we get

$$S_1 \ll T^{-\frac{1}{2}}\sum_{n\ll\sqrt{T}}\int_{\frac{1}{2}}^{1}\left(\frac{T}{n}\right)^{\sigma} d\sigma + \sum_{n\ll\sqrt{T}}\frac{1}{n}\int_{\frac{1}{2}}^{1} n^{\sigma}\, d\sigma + \int_{\frac{1}{2}}^{1} T^{\frac{\sigma}{2}-\frac{1}{2}}\, d\sigma \ll \sqrt{T}.$$

On the other hand, using 4.11.1 of Titchmarsh [9], we get

$$S_2 \ll \int_{1}^{a} T^{\sigma-\frac{1}{2}}\left|\sum_{n\ll T}\frac{1}{n^{\sigma+iT}} - \frac{(CT)^{1-\sigma-iT}}{1 - \sigma - iT} + O(T^{-\sigma})\right| d\sigma$$
$$\ll \sqrt{T}\sum_{n\ll T}\int_{1}^{a}\frac{1}{n^{\sigma}}\, d\sigma + \frac{1}{\sqrt{T}\log T} \ll \sqrt{T}. \qquad \text{Q.E.D.}$$

We shall next evaluate I_1. By the definition of $\xi(s)$, we get first

$$I_1 = \frac{1}{2\pi}\int_{C}^{T}\left(\frac{\zeta'}{\zeta}(a + it) + \left(\frac{1}{2}\psi\left(\frac{a + it}{2}\right) - \frac{1}{2}\log\pi\right)\right.$$
$$\left. + \frac{2(a + it) - 1}{(a + it)(a + it - 1)}\right)\zeta(a + i(t + \Delta))dt = J_1 + J_2 + J_3, \text{ say.}$$

It is easily seen that

$$J_1 = \frac{-1}{2\pi}\sum_{m=2}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda(m)}{m^a n^{a+i\Delta}}\int_{C}^{T}\frac{1}{(mn)^{it}}\, dt \ll \left|\frac{\zeta'}{\zeta}(a)\right|\left|\zeta(a)\right| \ll \frac{1}{\delta^2},$$

where $\Lambda(n)$ is the von-Mangoldt function.

$$J_2 = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n^{a+i\Delta}}\int_C^T \frac{1}{n^{it}}\left(\frac{\pi i}{2} - \log 2\pi + \log t + O\left(\frac{1}{t}\right)\right)dt$$

$$= \frac{1}{4\pi}\left(\frac{\pi i}{2} - \log 2\pi\right)T + \frac{1}{4\pi}(T\log T - T) + O\left(\frac{\log T}{\delta}\right).$$

By the functional equation, we get

$$\overline{I_2} = \frac{1}{2\pi i}\int_{a+iC}^{a+iT}\frac{\xi'}{\xi}(s)\frac{\zeta(s+i\Delta)}{\chi(s+i\Delta)}ds$$

$$= \frac{1}{2\pi i}\int_{a+iC}^{a+iT}\left(\frac{\zeta'}{\zeta}(s-i\Delta) + \left(\frac{1}{2}\psi\left(\frac{s-i\Delta}{2}\right) - \frac{1}{2}\log\pi\right)\right.$$

$$\left. + \frac{2(s-i\Delta)-1}{(s-i\Delta)(s-i\Delta-1)}\right)\zeta(s)\chi(1-s)ds + O(T^{a-\frac{1}{2}}\log^2 T)$$

$$= K_1 + K_2 + K_3 + O(T^{a-\frac{1}{2}}\log^2 T), \text{ say.}$$

By applying Lemma 5 of Gonek [7], we get

$$K_1 = \frac{1}{2\pi}\int_C^T \chi(1-a-it)\zeta(a+it)\frac{\zeta'}{\zeta}(a+i(t-\Delta))dt$$

$$= -M\left(\Delta, \frac{T}{2\pi}\right) + O(T^{a-\frac{1}{2}}),$$

where we put

$$M(\Delta, Y) = \sum_{1\le k\le Y}\sum_{m|k}\Lambda(m)m^{i\Delta}.$$

Similarly, we get

$$K_2 = \frac{\pi i}{4}\sum_{1\le n\le\frac{T}{2\pi}} 1 + \frac{1}{2}\sum_{1\le n\le\frac{T}{2\pi}}\log n + O(T^{a-\frac{1}{2}}\log T).$$

Since $J_3 \ll \frac{1}{\delta}\log T$ and $K_3 \ll T^{a-\frac{1}{2}}\log T$, we get

$$\sum_{0<\gamma\le T}\zeta(\rho+i\Delta) = -M\left(-\Delta, \frac{T}{2\pi}\right) + \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\sqrt{T}\log^2 T).$$

Finally, we get

$$M\left(\Delta, \frac{T}{2\pi}\right) = -\frac{\zeta'}{\zeta}(1-i\Delta)\frac{T}{2\pi} + \frac{\left(\frac{T}{2\pi}\right)^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta) + O(T\exp(-C\sqrt{\log T})),$$

because

$$\frac{1}{2\pi i}\int_{R_1} -\frac{\zeta'}{\zeta}(s-i\Delta)\zeta(s)\frac{\left(\frac{T}{2\pi}\right)^s}{s}ds = -\frac{\zeta'}{\zeta}(1-i\Delta)\frac{T}{2\pi} + \frac{\left(\frac{T}{2\pi}\right)^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta)$$

$$= M\left(\Delta, \frac{T}{2\pi}\right) + O(T\exp(-C\sqrt{\log T})),$$

where R_1 is the contour joining the points $b - iU$, $b + iU$, $\tilde{a} + iU$ and $\tilde{a} - iU$ and we put $U = \exp(C\sqrt{\log T})$, $\tilde{a} = 1 - \frac{C}{\log U}$ and $b = 1 + \frac{C}{\log T}$.

From these results we get our Theorem 3 as stated in the introduction.

§3. Proof of Theorem 4. We asssume R.H. in this section. We notice that the restriction on T imposed at the beginning of the preivious section is now harmless within the remainder term $O(T^{\varepsilon})$ for any positive ε. Now we have the following formula at hand;

$$\sum_{0<\gamma\leq T}\zeta\Big(\frac{1}{2}+i(\gamma+\Delta)\Big)=-M\Big(-\Delta,\frac{T}{2\pi}\Big)+\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\sqrt{T}\log^2 T).$$

We put for simplicity $X=T/2\pi$, $B=1+1/\log X$ and $A=1/\log U$, where U satisfies $X\leq U\leq X+1$ and $|U-\Delta-\gamma|\gg\frac{1}{\log U}$ for any γ. By our choice of parameters, we get first

$$Q\equiv\frac{1}{2\pi i}\int_{B-iU}^{B+iU}\Big(-\frac{\zeta'}{\zeta}(s-i\Delta)\Big)\zeta(s)\frac{X^s}{s}ds=M(\Delta,X)+O(\log^2 X).$$

On the other hand, we have

$$Q=-\frac{1}{2\pi i}\Big(\int_{B+iU}^{-A+iU}+\int_{-A-iU}^{B-iU}-\int_{-A-iU}^{-A+iU}\Big)\Big(-\frac{\zeta'}{\zeta}(s-i\Delta)\Big)\zeta(s)\frac{X^s}{s}ds$$
$$-\frac{\zeta'}{\zeta}(1-i\Delta)X+\frac{X^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta)-\sum_{|\gamma|\leq U}\frac{\zeta(\rho+i\Delta)X^{\rho+i\Delta}}{\rho+i\Delta}+O(1)$$
$$=Q_1+Q_2+Q_3-\frac{\zeta'}{\zeta}(1-i\Delta)X+\frac{X^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta)+Q_4+O(1),\text{ say.}$$

We get as in the previous section

$$Q_1\ll\frac{X}{U}\Big(\int_1^B+\int_0^1+\int_{-A}^0\Big)\Big|\frac{\zeta'}{\zeta}(\sigma+i(U-\Delta))\Big|\,|\zeta(\sigma+iU)|\,d\sigma$$
$$\ll\int_0^1\Big(\sum_{|U-\Delta-\gamma|\leq 1}\frac{1}{|\sigma+i(U-\Delta)-\rho|}+\log U\Big)|\zeta(\sigma+iU)|\,d\sigma+\sqrt{X}\log X$$
$$\ll\log U\int_{\frac{1}{2}+\frac{C}{\log U}}^{1}\frac{U^{\sigma-\frac{1}{2}}}{\sigma-1/2}\Big(\sum_{n\ll\sqrt{U}}\frac{1}{n^{\sigma}}+U^{\frac{1}{2}-\sigma}\sum_{n\ll\sqrt{U}}\frac{1}{n^{1-\sigma}}\Big)d\sigma$$
$$+\log^2 U\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{C}{\log U}}\sum\nolimits_{n\ll\sqrt{U}}\frac{1}{\sqrt{n}}d\sigma+\sqrt{X}\log X\ll\sqrt{X}\log X.$$

Similarly, we get

$$Q_2\ll\sqrt{X}\log X.$$

We shall next estimate Q_3. First we have

$$Q_3=\frac{1}{2\pi i}\int_{-A-iU}^{-A+iU}\Big(\frac{\zeta'}{\zeta}(1-s+i\Delta)-\frac{\chi'}{\chi}(1-s+i\Delta)\Big)\zeta(1-s)\chi(s)\frac{X^s}{s}ds$$
$$=Q_8+Q_9,\text{ say.}$$

$$Q_8=-X^{-A}\frac{1}{2\pi}\sum_{m=2}^{\infty}\frac{\Lambda(m)}{m^{1+A+i\Delta}}\sum_{n=1}^{\infty}\frac{1}{n^{1+A}}\Big(\int_1^U+\int_{-U}^{-1}\Big)(Xmn)^{it}\frac{\chi(-A+it)}{-A+it}dt$$
$$+O(\log^2 U)$$
$$=Q_8'+Q''_8+O(\log^2 U)\text{ say.}$$

$$Q_8'\ll X^{-A}\sum_{m=2}^{\infty}\frac{\Lambda(m)}{m^{1+A}}\sum_{n=1}^{\infty}\frac{1}{n^{1+A}}\Big|\int_1^U(e^{-it\log\frac{t}{2\pi eXmn}}t^{A-\frac{1}{2}}+O(t^{A-\frac{3}{2}}))dt\Big|$$
$$\ll\log^2 U,$$

where we have used Lemma 4.5 of Titchmarsh [9] to estimate the last integral.

Treating Q''_8 and Q_9 similarly, we get

$$Q_3\ll\log^2 U.$$

Finally, since

$$Q_4 \ll \sqrt{X}\Big(\sum_{0<\gamma\ll X} \frac{\left|\zeta\left(\frac{1}{2}+i(\gamma+\Delta)\right)\right|^2}{\gamma}\Big)^{\frac{1}{2}}\Big(\sum_{0<\gamma\ll X}\frac{1}{\gamma}\Big)^{\frac{1}{2}} \ll \sqrt{X}\log^{\frac{5}{2}} X,$$

we get

$$M(\Delta, X) = -\frac{\zeta'}{\zeta}(1-i\Delta)X + \frac{X^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta) + O(\sqrt{X}\log^{\frac{5}{2}} X).$$

This proves our Theorem 4.

§4. Proof of Theorems 1 and 2. By evaluating $\frac{1}{2\pi i}\int_R \frac{\xi'}{\xi}(s)\,\zeta'(s)ds$, we get as in the preceding section

$$\sum_{0<\gamma\le T}\zeta'(\rho) = -\sum_{mn\le\frac{T}{2\pi}}\Lambda(m)\log n + \sum_{1\le n\le\frac{T}{2\pi}}\log^2 n + O(T^{a-\frac{1}{2}}\log^3 T),$$

where we use the following lemma with $a = 1 + 1/\log T$ which can be proved in the same manner as Lemma 1.

Lemma 2. $\int_{1-a}^{a} |\zeta'(\sigma+iT)|\,d\sigma \ll \sqrt{T}\log T.$

By evaluating $\frac{1}{2\pi i}\int_{R_1}\left(-\frac{\zeta'}{\zeta}(s)\right)(-\zeta'(s))\frac{(T/2\pi)^s}{s}ds$, we get as in the section 2,

$$\sum_{mn\le\frac{T}{2\pi}}\Lambda(m)\log n = \frac{1}{4\pi}T\log^2\frac{T}{2\pi} - (C_o+1)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+ (2 - C_1 + C_o)\frac{T}{2\pi} + O(T\exp(-C\sqrt{\log T})).$$

This proves Theorem 1.

If we assume the Riemann Hypothesis, it is clear as in the previous section that the remainder term $O(T\exp(-C\sqrt{\log T}))$ can be replaced by $O(\sqrt{T}\log^{\frac{7}{2}} T)$. This proves our Theorem 2.

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