82. *Spined Products of Some Semigroups**)

By Miroslav ĆIRIĆ and Stojan BOGDANOVIĆ

University of Niš, Yugoslavia

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1993)

Spined products of semigroups were first defined and studied by N. Kimura, 1958, [7]. After that, spined products have been considered many a time, predominantly those of a band and a semilattice of semigroups with respect to their common semilattice homomorphic image. Spined and subdirect products of a band and a semilattice of groups are studied by M. Yamada [13], [14], J. M. Howie and G. Lallement [6] and by M. Petrich [10]; spined products of a band and some types of semilattices of monoids are studied by F. Pastijn [8], A. El-Qallali [3], [4], and by R. J. Warne [12]. For other considerations of these products, we refer to [4], [5], [7], [9], [15]. In the quoted papers, spined products are considered in connection with some types of bands of semigroups. In this paper, we give a general composition for bands of semigroups that are (punched) spined products of a band and a semilattice of semigroups. This composition, in some sense, is a generalization of a well-known semilattice composition (see Theorem III 7.2. [9]).

Let B be a band. By $\leq_1$ and $\leq_2$ we denote quasi-orders on B defined by $i \leq_1 j \Leftrightarrow ij = j$, $i \leq_2 j \Leftrightarrow ji = j$, and by $\leq$ we denote the *natural order* on B defined by "$i \leq j$ means that $i \leq_1 j$ and $i \leq_2 j$". For $i \in B$, we will denote by $[i]$ the class of an element i in the greatest semilattice decomposition of a band B (so $[i]$ is an element of the greatest semilattice homomorphic image of B). If S is a band B of semigroups S_i, $i \in B$, then for $k \in B$, F_k will denote the semigroup $F_k = \cup\{S_i \mid i \in B, [i] \geq [k]\}$. If θ is a homomorphism of a semigroup S into a semigroup S', and if T is a common subsemigroup of S and S', then θ is a *T-homomorphism* if $a\theta = a$, for all $a \in T$. A subsemigroup T of a semigroup S is a *retract* of S if there exists a homomorphism θ of S onto T such that $a\theta = a$, for all $a \in T$. We call such a homomorphism a *retraction*. If T is a subsemigroup of a semigroup S, then we say that S is an *oversemigroup* of T. If ρ is a congruence on a semigroup S, then we denote by $\rho^\natural$ the natural homomorphism of S onto S/ρ. If P and Q are two semigroups having a common homomorphic image Y, then the *spined product of P and Q with respect to Y* is $S = \{(a, b) \in P \times Q \mid a\varphi = b\psi\}$, where $\varphi : P \to Y$ and $\psi : Q \to Y$ are homomorphisms onto Y. If Y is a semilattice and P and Q are a semilattice Y of semigroups P_α, $\alpha \in Y$, and Q_α, $\alpha \in Y$, respectively, then the spined product of P and Q with respect to Y is $S = \cup_{\alpha \in Y} P_\alpha \times Q_\alpha$. A subsemigroup S of a spined product of semigroups P and Q with respect to Y, that is also a subdirect product of P and Q, is a *punched spined product of P and Q with respect to Y.*

*) Supported by Grant 0401A of RFNS through Math. Inst. SANU.

For undefined notions and notations we refer to [5] and [9].

Lemma 1. *Let B be a band. To each $i \in B$ we associate a semigroup S_i and an oversemigroup D_i of S_i such that $D_i \cap D_j = \emptyset$, if $i \neq j$. For $i, j \in B$, $[i] \geq [j]$, let $\phi_{i,j}$ be a mapping of S_i into D_i and suppose that the family of $\phi_{i,j}$ satisfies the following conditions:*

(1) *$\phi_{i,i}$ is the identity mapping on S_i, for every $i \in B$;*

(2) *$(S_i\phi_{i,ij})(S_j\phi_{j,ij}) \subseteq S_{ij}$, for all $i, j \in B$;*

(3) *$[(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,k} = (a\phi_{i,k})(b\phi_{j,k})$, for $a \in S_i$, $b \in S_j$, $[ij] \geq [k]$, $i, j, k \in B$.*

Define a multiplication $*$ on $S = \cup_{i\in B} S_i$ by: $a * b = (a\phi_{i,ij})(b\phi_{j,ij})$, for $a \in S_i$, $b \in S_j$. Then S is a band B of semigroups S_i, $i \in B$, in notation $S = (B\,;\, S_i, \phi_{i,j}, D_i)$.

Proof. Assume $a \in S_i$, $b \in S_j$, $c \in S_k$, $i, j, k \in B$. Then by (3) we have

$$\begin{aligned}(a*b)*c &= [(a\phi_{i,ij})(b\phi_{j,ij})]*c = [(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,ijk}(c\phi_{k,ijk}) \\ &= (a\phi_{i,ijk})(b\phi_{j,ijk})(c\phi_{k,ijk}) = (a\phi_{i,ijk})[(b\phi_{j,jk})(c\phi_{k,jk})]\phi_{jk,ijk} \\ &= a*[(b\phi_{j,jk})(c\phi_{k,jk})] = a*(b*c).\end{aligned}$$

Thus, S is a semigroup. Clearly, it is a band B of semigroups S_i.

If we assume $i = j$ in (3), then we obtain that $\phi_{i,k}$ is a homomorphism, for all $i, k \in B$, $[i] \geq [k]$. If $D_i = S_i$, for each $i \in B$, then we write $S = (B\,;\, S_i, \phi_{i,j})$. Here the condition (2) can be omitted.

Theorem 1. *Let S be a band B of semigroups S_i, $i \in B$. Then*

(a) *$S = (B\,;\, S_i, \phi_{i,j}, D_i)$ if and only if for every $k \in B$ there exists an oversemigroup D_k of S_k and an S_k-homomorphism of F_k into D_k;*

(b) *if $S = (B\,;\, S_i, \phi_{i,j}, D_i)$, then we can assume that $D_k = \{a\phi_{i,k} \mid a \in S_i, [i] \geq [k]\}$, for each $k \in B$;*

(c) *$S = (B\,;\, S_i, \phi_{i,j})$ if and only if for every $k \in B$, S_k is a retract of F_k.*

Proof. (a) If $S = (B\,;\, S_i, \phi_{i,j}, D_i)$, then for $k \in B$, the mapping $\theta_k : F_k \to D_k$ defined by: $a\theta_k = a\phi_{i,k}$, for $a \in S_i$, $[i] \geq [k]$, is an S_k-homomorphism.

Conversely, suppose that for every $k \in B$ there exists an oversemigroup D_k and an S_k-homomorphism θ_k of F_k into D_k. For $i, j \in B$, $[i] \geq [j]$, define a mapping $\phi_{i,j}$ of S_i into D_j by: $a\phi_{i,j} = a\theta_j$, $a \in S_i$. It is clear that (1) holds. Let $a \in S_i$, $b \in S_j$, $i, j \in B$. Then $a, b \in F_{ij}$, $ab \in S_{ij}$, whence $(a\phi_{i,ij})(b\phi_{j,ij}) = (a\theta_{ij})(b\theta_{ij}) = (ab)\theta_{ij} = ab$. Let $k \in B$, $[ij] \geq [k]$. Then $[(a\phi_{i,ij})(b\phi_{j,ij})]\phi_{ij,k} = (ab)\theta_k = (a\theta_k)(b\theta_k) = (a\phi_{i,k})(b\phi_{j,k})$. Thus, $S = (B\,;\, S_i, \phi_{i,j}, D_i)$.

(b). In notations from (a), for $k \in B$, $\{a\phi_{i,k} \mid a \in S_i, [i] \geq [k]\} = F_k\phi_k$, and it is a subsemigroup of D_k. Clearly, every one of the conditions (1)-(3) of Lemma 1 holds for D_k if and only if it holds for $F_k\phi_k$. Thus, (b) holds.

(c) This follows by (a).

If B is a semilattice, then $S = (B\,;\, S_i, \phi_{i,j}, D_i)$ is a semigroup constructed as in Theorem III 7.2. [9]. In this case, for each $k \in B$, S_k is an ideal of F_k, so using well known results from the theory of ideal extensions of semigroups, in Theorem III 7.2. [9] was proved that *every semigroup S that it a semilattice Y of semigroups S_α, $\alpha \in Y$, can be composed as $S = (Y ; S_\alpha, \phi_{\alpha,\beta}, D_\alpha)$, and, furthermore, each D_α can be chosen to be a dense extension of S_α and that $D_\alpha = \{b\phi_{\beta,\alpha}\, \beta \geq \alpha,\ b \in S_\beta\}$*. This fact will be used in the next considerations to representing a semilattice of arbitrary semigroups.

Also, we will give another construction. If $S = (B ; S_i, \phi_{i,j}, D_i)$ and if

(4) $S_i\phi_{i,j} \subseteq S_j$, for $[i] = [j]$, $i, j \in B$;

(5) $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$, for $[i] = [j] \geq [k]$, $i, j, k \in B$;

then we will write $S = [\![B ; S_i, \phi_{i,j}, D_i]\!]$. If $S = (B ; S_i, \phi_{i,j})$ with (4) and (5), then we write $S = [\![B ; S_i, \phi_{i,j}]\!]$. If $S = (B ; S_i, \phi_{i,j})$ and if $\{\phi_{i,j} \mid i, j \in B,\ [i] \geq [j]\}$ is a *transitive system of homomorphisms*, i.e. if $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$, for $[i] \geq [j] \geq [k]$, then we will write $S = [B ; S_i, \phi_{i,j}]$.

Let B be a band. To each $i \in B$ we associate a semigroup S_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $\varphi_{i,j}$ and $\psi_{i,j}$ be homomorphisms of S_i into S_j defined for $i \geq_1 j$ and $i \geq_2 j$, respectively, such that:

(6) for every $i \in B$, $\varphi_{i,i} = \psi_{i,i}$ is the identity mapping on S_i;

(7) $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$, for $i \geq_1 j \geq_1 k$;

(8) $\psi_{i,j}\psi_{j,k} = \psi_{i,k}$, for $i \geq_2 j \geq_2 k$;

(9) $\varphi_{i,j}\psi_{j,kj} = \psi_{i,k}\varphi_{k,kj}$, for $i \geq_1 j$, $i \geq_2 k$.

Define a multiplication $*$ on $S = \cup_{i \in B} S_i$ by: $a * b = (a\varphi_{i,ij})(b\psi_{j,ij})$, for $a \in S_i$, $b \in S_j$, $i,j \in B$. Then by [11] S is a band B of semigroups S_i, $i \in B$. This construction is introduced by B. M. Schein [11], and it has been explored by the authors in [1], where it is denoted by $S = [B ; S_i, \varphi_{i,j}, \psi_{i,j}]$ and called a *strong band of semigroups S_i*. It is easy to prove the following lemma:

Lemma 2. *If $S = [B ; S_i, \phi_{i,j}]$, then $S = [B ; S_i, \varphi_{i,j}, \psi_{i,j}]$, where $\varphi_{i,j} = \phi_{i,j}$, for $i \geq_1 j$, and $\psi_{i,j} = \phi_{i,j}$, for $i \geq_2 j$. Conversely, if $S = [B ; S_i, \varphi_{i,j}, \psi_{i,j}]$, then $S = [B ; S_i, \phi_{i,j}]$, where $\phi_{i,j} = \varphi_{i,ij}\psi_{ij,j}$, for $[i] \geq [j]$.*

Therefore, the constructions $[B ; S_i, \varphi_{i,j}, \psi_{i,j}]$ and $[B ; S_i, \phi_{i,j}]$ are equivalent. So $[B ; S_i, \phi_{i,j}]$ will be called a *strong band of semigroups S_i*. If B is a semilattice, then we obtain a well known *strong semilattice of semigroups.*

The following lemma is proved by B. M. Schein [11], in the case when S_i are monoids, and it is immediate to extend this proof to the general case.

Lemma 3. *Let B be a rectangular band.*

If $S = [B ; S_i, \phi_{i,j}]$, then each $\phi_{i,j}$, is an isomorphism of S_i onto S_j, $i, j \in B$, and for every $k \in B$, the mapping θ of S into $S_k \times B$ defined by $a\theta = (a\phi_{i,k}, i)$, for $a \in S_i$, $i \in B$, is an isomorphism.

Conversely, if $S = T \times B$, if we assume that $S_i = T \times \{i\}$, $i \in B$ and if we assume that $\phi_{i,j}$ is a mapping of S_i into S_j, $i, j \in B$, defined by $(a, i)\phi_{i,j} = (a, j)$, $a \in S_i$, then $S = [B ; S_i, \phi_{i,j}]$.

Theorem 2. *Let a band B be a semilattice Y of rectangular bands B_α. If $S = (B ; S_i, \phi_{i,j}, D_i)$, then*

(A1) *S is a semilattice Y of semigroups $S_\alpha = (B_\alpha ; S_i, \phi_{i,j}, D_i)$, $\alpha \in Y$;*

(A2) *a relation ρ on S defined by: $a\rho b$ if and only if $a \in S_i$, $b \in S_j$, $[i] = [j] = \alpha$, and $a\phi_{i,k} = b\phi_{j,k}$ for all $k \in B$, $\alpha \geq [k]$, is a congruence on S and T*

$= S/\rho$ *is a semilattice* Y *of semigroups* $T_\alpha = S_\alpha\rho^\natural$;

(A3) S *is a punched spined product of* T *and* B *with respect to* Y.

Conversely, if S *is a punched spined product of* $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta},\ D_\alpha)$ *and* B *with respect to* Y *and if we assume that*:

(B1) $S_i = (T_\alpha \times \{i\}) \cap S,\ D_i = D_\alpha \times \{i\}$, *for* $i \in B_\alpha$;

(B2) *for* $i, j \in B$, $[i] \geq [j]$, *a mapping* $\phi_{i,j}$ *of* S_i *into* D_j *is defined by*:

$$(a, i)\phi_{i,j} = (a\phi_{[i],[j]}, j)\ ;$$

then $S = (B\ ;\ S_i,\ \phi_{i,j},\ D_i)$.

Proof. Let $S = (B\ ;\ S_i,\ \phi_{i,j},\ D_i)$. Then it is clear that (A1) holds.

(A2) It is clear that ρ is an equivalence relation. Assume that $a\rho b$ and $x \in S$. Let $a \in S_i$, $b \in S_j$, $i, j \in B_\alpha$, $\alpha \in Y$ and let $x \in S_k$, $k \in B_\beta$, $\beta \in Y$. Then $ax \in S_{ik}$, $bx \in S_{jk}$, $ik, jk \in B_{\alpha\beta}$. Assume $l \in B$, $\alpha\beta \geq [l]$. Then $\alpha \geq [l]$, so $a\phi_{i,l} = b\phi_{j,l}$. By (3) we obtain that

$$\begin{aligned}(ax)\phi_{ik,l} &= [(a\phi_{i,ik})(x\phi_{k,ik})]\phi_{ik,l} = (a\phi_{i,l})(x\phi_{k,l}) = (b\phi_{j,l})(x\phi_{k,l}) \\ &= [(b\phi_{j,jk})(x\phi_{k,jk})]\phi_{jk,l} = (bx)\phi_{jk,l}.\end{aligned}$$

Thus, $ax\rho bx$. Similarly we prove that $xa\rho xb$. Therefore, ρ is a congruence. Let σ be a semilattice congruence on S determined by the partition $\{S_\alpha \mid \alpha \in Y\}$. Then $\rho \subseteq \sigma$, so $T = S/\rho$ is a semilattice Y of semigroups $T_\alpha = S_\alpha\rho^\natural$.

(A3) Let ξ be the band congruence on S determined by the partition $\{S_i \mid i \in B\}$. Clearly, $\rho \cap \xi = \varepsilon$, where ε is the equality relation on S, so S is a subdirect product of T and B, where a one-to-one homomorphism Φ of S into $T \times B$ is given by $a\Phi = (a\rho, a\xi)$, $a \in S$. Assume $a \in S$. Let $a \in S_i$, $i \in B_\alpha$, $\alpha \in Y$. Then $a \in S_\alpha$, so $a\rho \in T_\alpha$, and $a\xi = i \in B_\alpha$. Thus, $S\Phi \subseteq \cup_{\alpha\in Y} T_\alpha \times B_\alpha$, so S is a punched spined product of T and B.

Conversely, let $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta},\ D_\alpha)$, let S be a punched spined product of T and B and let S_i, D_i and $\phi_{i,j}$ be defined by (B1) and (B2). Then it is not hard to verify that $S = (B\ ;\ S_i,\ \phi_{i,j},\ D_i)$.

Theorem 3. *Let a band* B *be a semilattice* Y *of rectangular bands* B_α. *If* $S = [\![B\ ;\ S_i,\ \varphi_{i,j},\ D_i]\!]$, *then*

(C1) S *is a semilattice* Y *of semigroups* $S_\alpha = [B_\alpha\ ;\ S_i,\ \phi_{i,j}]$, $\alpha \in Y$;

(C2) *each* S_α *is isomorphic to* $T_\alpha \times B_\alpha$, *where* T_α *is a semigroups isomorphic to each* S_i, $i \in B_\alpha$;

(C3) *there exists a semilattice composition* $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta},\ D_\alpha)$ *such that* S *is isomorphic to the spined product of* B *and* T *with respect to* Y. *Furthermore, if* $S = [\![B\ ;\ S_i,\ \phi_{i,j}]\!]$ $(S = [B\ ;\ S_i,\ \phi_{i,j}])$, *then* T *can be chosen to* $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta})$ $(T = [Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta}])$.

Conversely, if S *is a spined product of* $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta},\ D_\alpha)$ *and* B *with respect to* Y *and if we assume that*:

(D1) $S_i = T_\alpha \times \{i\}$, $D_i = D_\alpha \times \{i\}$, *for* $i \in B_\alpha$;

(D2) *for* $i, j \in B$, $[i] \geq [j]$, *a mapping* $\phi_{i,j}$ *of* S_i *into* D_i *is defined by*:

$$(a, j)\phi_{i,j} = (a\phi_{[i],[j]}, j)\ ;$$

then $S = [\![B\ ;\ S_i,\ \phi_{i,j},\ D_i]\!]$. *Furthermore, if* $T = (Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta})$ $(T = [Y\ ;\ T_\alpha,\ \phi_{\alpha,\beta}])$, *then* $S = [\![B\ ;\ S_i,\ \phi_{i,j}]\!]$ $(S = [B\ ;\ S_i,\ \phi_{i,j}])$.

Proof. By (5) and by Lemma 3 it follows that (C1) and (C2) hold.

For any $\alpha \in Y$, fix $0_\alpha \in B_\alpha$, and assume that $T_\alpha = S_{0_\alpha}$, $D_\alpha = D_{0_\alpha}$. For

$\alpha, \beta \in Y$, $\alpha \geq \beta$, define a mapping $\phi_{\alpha,\beta}$ of T_α into D_β by $\phi_{\alpha,\beta} = \phi_{0_\alpha,0_\beta}$. It is clear that $\phi_{\alpha,\alpha}$ is the identity map of T_α, for any $\alpha \in Y$. Assume $\alpha, \beta \in Y$, $a \in T_\alpha$, $b \in T_\beta$. Then by (3) we have that

$$(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) = (a\phi_{0_\alpha,0_{\alpha\beta}})(b\phi_{0_\beta,0_{\alpha\beta}}) = [(a\phi_{0_\alpha,0_\alpha 0_\beta})(b\phi_{0_\beta,0_\alpha 0_\beta})]\phi_{0_\alpha 0_\beta,0_{\alpha\beta}},$$

so by (2) and (4) we obtain that $(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) \in S_{0_{\alpha\beta}} = T_{\alpha\beta}$, whence it follows that $(T_\alpha\phi_{\alpha,\alpha\beta})(T_\beta\phi_{\beta,\alpha\beta}) \subseteq T_{\alpha\beta}$. For $\gamma \in Y$, $\alpha\beta \geq \gamma$, by (3) and (5)

$$\begin{aligned}[(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})]\phi_{\alpha\beta,\gamma} &= [(a\phi_{0_\alpha,0_{\alpha\beta}})(b\phi_{0_\beta,0_{\alpha\beta}})]\phi_{0_{\alpha\beta},0_\gamma}\\ &= [(a\phi_{0_\alpha,0_\alpha 0_\beta})(b\phi_{0_\beta,0_\alpha 0_\beta})]\phi_{0_\alpha 0_\beta,0_{\alpha\beta}}\phi_{0_{\alpha\beta},0_\gamma} = [(a\phi_{0_\alpha,0_\alpha 0_\beta})(b\phi_{0_\beta,0_\alpha 0_\beta})]\phi_{0_\alpha 0_\beta,0_\gamma}\\ &= (a\phi_{0_\alpha,0_\gamma})(b\phi_{0_\beta,0_\gamma}) = (a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}).\end{aligned}$$

Thus, by Lemma 1, there exists a semilattice composition $S = (Y\ ;\ T_\alpha, \phi_{\alpha,\beta}, D_\alpha)$.

Define a mapping Φ of S into $T \times B$ by: $a\Phi = (a\phi_{i_\alpha,0_\alpha}, i_\alpha)$, if $a \in S_{i_\alpha}$, $i_\alpha \in B_\alpha$, $\alpha \in Y$. Clearly, $S\Phi \subseteq \cup_{\alpha\in Y} T_\alpha \times B_\alpha$. Since $\phi_{i_\alpha,0_\alpha}$ is an isomorphism of S_{i_α} onto S_{0_α} (by Lemma 3), then Φ is a bijection of S onto $\cup_{\alpha\in Y} T_\alpha \times B_\alpha$.

Assume $a \in S_{i_\alpha}$, $b \in S_{i_\beta}$, $i_\alpha \in B_\alpha$, $i_\beta \in B_\beta$, $\alpha, \beta \in Y$. Then by (5) and (3)

$$\begin{aligned}(a\Phi)(b\Phi) &= (a\phi_{i_\alpha,0_\alpha}, i_\alpha)(b\phi_{i_\beta,0_\beta}, i_\beta) = ((a\phi_{i_\alpha,0_\alpha}\phi_{\alpha,\alpha\beta})(b\phi_{i_\beta,0_\beta}\phi_{\beta,\alpha\beta}), i_\alpha i_\beta)\\ &= ((a\phi_{i_\alpha,0_\alpha}\phi_{0_\alpha,0_{\alpha\beta}})(b\phi_{i_\beta,0_\beta}\phi_{0_\beta,0_{\alpha\beta}}), i_\alpha i_\beta) = ((a\phi_{i_\alpha,0_{\alpha\beta}})(b\phi_{i_\beta,0_{\alpha\beta}}), i_\alpha i_\beta)\\ &= ([(a\phi_{i_\alpha,i_\alpha i_\beta})(b\phi_{i_\beta,i_\alpha i_\beta})]\phi_{i_\alpha i_\beta 0_{\alpha\beta}}, i_\alpha i_\beta) = [(a\phi_{i_\alpha,i_\alpha i_\beta})(b\phi_{i_\beta,i_\alpha i_\beta})]\Phi = (ab)\Phi.\end{aligned}$$

Thus, Φ is an isomorphism of S onto $\cup_{\alpha\in Y} T_\alpha \times B_\alpha$ and (C3) holds. The rest is obvious.

Conversely, let $T = (Y\ ;\ S_\alpha, \phi_{\alpha,\beta}, D_\alpha)$, let S be a spined product of T and B with respect to Y, and assume that S_i, D_i and $\phi_{i,j}$ are defined by (D1) and (D2). Then by Theorem 2. we obtain that $S = (B\ ;\ S_i, \phi_{i,j}, D_i)$. It is clear that (4) holds. Assume $i, j, k \in B$, $[i] = [j] \geq [k]$, let $[i] = [j] = \alpha$, $[k] = \beta$, and let $(a, i) \in S_i$. Then $(a, i)\phi_{i,j}\phi_{j,k} = (a\phi_{\alpha,\alpha}\phi_{\alpha,\beta}, k) = (a\phi_{\alpha,\beta}, k) = (a, i)\phi_{i,k}$. Therefore, (5) holds. Hence, $S = [\![B\ ;\ S_i, \phi_{i,j}, D_i]\!]$. The rest is obvious.

In the next considerations we will assume that S is a band B of monoids S_i, $i \in B$, that B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$. For $i \in B$, let e_i denote the identity element of S_i. We will give some applications of the previous results to bands of monoids. If $S = (B\ ;\ S_i, \phi_{i,j})$, then it is easy to verify that *$\phi_{i,j}$ are uniquely determined by*: $a\phi_{i,j} = e_j a e_j$, $a \in S_i$, $i, k \in B$, $[i] \geq [k]$. Thus, *$S = (B\ ;\ S_i, \phi_{i,j})$ if and only if for every $k \in B$, the mapping $\phi_k : F_k \to S_k$, defined by $a\phi_k = e_k a e_k$, $a \in F_k$, is a homomorphism.* If $\{e_i \mid i \in B\}$ is a subsemigroup of S, then S is a *proper band of monoids* S_i, [11]. If for every $\alpha \in Y$, $\{e_i \mid i \in B_\alpha\}$ is a subsemigroup, then S is a *semiproper band of monoids* S_i. It is not hard to prove that *S is a semiproper band of monoids S_i if and only if $S = (B\ ;\ S_i, \phi_{i,j})$ and $\phi_j\phi_k = \phi_k$, for $j, k \in B$, $[j] = [k]$*. Also, *S is a spined product of a band and a semilattice of monoids if and only if S is a semiproper band of monoids and $a\phi_j\phi_k = a\phi_k$, for all $a \in S_\alpha$, $j, k \in B$, $[j] = \alpha \geq [k]$*. Using these facts and using Theorem 2 [11], we obtain

Corollary 1. *A semigroup S is a strong (proper) band of monoids if and only if S is a spined product of a band and a strong (proper) semilattice of* *monoids.*

For proper bands of monoids, the previous corollary is proved by F. Pastijn [8].

Corollary 2. $S=(B\,;S_i,\phi_{i,j})$, *where* S_i *are unipotent monoids, if and only if* S *is a spined product of a band and a semilattice of unipotent monoids.*

Spined products of a band and a semilattice of cancellative (therefore, unipotent) monoids are considered by R. J. Warne [12] and by A. El-Qallali [3], [4].

Corollary 3. *The following conditions on a semigroup* S *are equivalent*:

(i) S *is an orthodox band of groups*;

(ii) S *is regular and a subdirect product of a band and a semilattice of groups*;

(iii) S *is a spined product of a band and a semilattice of groups.*

M. Yamada [13] proved (i) ⇔ (iii) and M. Petrich [10] proved (i) ⇔ (ii).

References

[1] M. Ćirić and S. Bogdanović: Sturdy bands of semigroups. Collect. Math. Barcelona, **41** (3), 189–195 (1990).

[2] ——: Normal band compositions of semigroups (to appear).

[3] A. El-Qallali: Left regular bands of groups of left quotients. Glasgow Math. J., **33**, 29-40 (1991).

[4] ——: $\mathcal{L}^*$-unipotent semigroups. J. Pure Appl. Algebra, **62**, 19–33 (1989).

[5] J. M. Howie: An Introduction to Semigroup Theory. Academic Press (1976).

[6] J. M. Howie and G. Lallement: Certain fundamental congruences on a regular semigroup. Proc. Glasgow Math. Assoc., **7**, 145–159 (1966).

[7] N. Kimura: The structure of idempotent semigroups. I. Pacific J. Math., **8**, 257–275 (1958).

[8] F. Pastijn: On Schein's structure theorem on proper bands of monoids. Teor. Polugrupp Prilozh., **5**, 82–86 (1985) (Russian).

[9] M. Petrich: Introduction to Semigroups. Merill, Ohio (1973).

[10] ——: Regular semigroups which are subdirect products of a band and a semilattice of groups. Glasgow Math. J., **14**, 27–49 (1973).

[11] B. M. Schein: Bands of monoids. Acta Sci. Math. Szeged, **36**, 145–154 (1974).

[12] R. J. Warne: TC semigroups and related semigroups. Workshop on semigroups, formal languages and combinatorics words (August 29–31, 1992), Abstracts, pp. 130–132.

[13] M. Yamada: Strictly inversive semigroups. Bull. Shimane Univ., **13**, 128–138 (1964).

[14] ——: Inversive semigroups. III. Proc. Japan. Acad., **41**, 221–224 (1965).

[15] ——: Regular semigroups whose idempotents satisfies permutation identities. Pacific J. Math., **21**, 371–392 (1967).

[16] M. Yamada and N. Kimura: Note on idempotent semigroups. II. Proc. Japan Acad., **34**, 110–112 (1958).