75. *Meromorphic Solutions of Some Second Order Differential Equations*

By Katsuya ISHIZAKI

Department of Mathematics, Tokyo National College of Technology

(Communicated by Kiyosi ITÔ, M. J. A., Oct. 12, 1993)

1. Introduction. In this note, we investigate the relation between meromorphic solutions of a Riccati equation

$$u' + u^2 + A(z) = 0 \tag{1.1}$$

and meromorphic solutions of some second order differential equation

$$\varphi'' + 3\varphi'\varphi + \varphi^3 + 4A(z)\varphi + 2A'(z) = 0, \tag{1.2}$$

where $A(z)$ is a meromorphic function.

For any solutions $u_1(z)$, $u_2(z)$ of (1.1), $\varphi(z) := u_1(z) + u_2(z)$ satisfies the equation (1.2). In fact, denoting by $\Phi(z, \varphi)$ the left-hand side of (1.2), we have

$$\Phi(z, \varphi) = 3u_1U_1(z, u_2) + 3u_2U_1(z, u_1) + U_2(z, u_1) + U_2(z, u_2), \tag{1.3}$$

where $U_1(z, u) = u' + u^2 + A(z)$, $U_2(z, u) = u'' + 3u'u + u^3 + A(z)u + A'(z) = \dfrac{dU_1(z, u(z))}{dz} + uU_1(z, u)$.

It is easy to see that if $u(z)$ satisfies the equation (1.1), then $U_j(z, u(z)) = 0$, $j = 1, 2$. This means that sum $\varphi(z)$ of solutions $u_1(z)$, $u_2(z)$ of the equation (1.1) is a solution of the equation (1.2). Conversely, we get the following theorems:

Theorem 1.1. *Suppose that $A(z)$ is an entire function. Then the equation* (1.2) *admits a meromorphic solution $\varphi(z)$. Moreover, for any meromorphic solution $\varphi(z)$ of* (1.2), *there exist meromorphic solutions $u_1(z)$, $u_2(z)$ of the Riccati equation* (1.1) *such that $\varphi(z) = u_1(z) + u_2(z)$.*

In this note, we use standard notations in the Nevanlinna theory (see, e.g., [3], [6], [7]). Let $f(z)$ be a meromorphic function. As usual, $m(r, f)$, $N(r, f)$, and $T(r, f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. A function $\varphi(r)$, $0 \leqq r < \infty$, is said to be $S(r, f)$ if there is a set $E \subset \boldsymbol{R}^+$ of finite linear measure such that $\varphi(r) = o(T(r, f))$ as $r \to \infty$ with $r \notin E$. We say that meromorphic solutions $u(z)$ and $\varphi(z)$ are admissible solutions (1.1) and (1.2), if $T(r, A) = S(r, u)$ and $T(r, A) = S(r, \varphi)$, respectively. For some property P, we denote by $n_{\mathrm{P}}(r, c; f)$ the number of c-points in $|z| \leqq r$ that admit the property P. The integrated counting function $N_{\mathrm{P}}(r, c; f)$ is defined in the usual fashion. Suppose $N(r, c; f) \neq S(r, f)$ for a $c \in \boldsymbol{C} \cup \{\infty\}$. If $N(r, c; f) - N_{\mathrm{P}}(r, c; f) = S(r, f)$, then we say that *almost all* c-points admit the property P.

Theorem 1.2. *Suppose that the equations* (1.1) *and* (1.2) *possess an admissible solution $u_1(z)$ and a meromorphic solution $\varphi(z)$, respectively. If*

$u_1(z)$ and $\varphi(z)$ share almost all poles, then the function $u_2(z) := \varphi(z) - u_1(z)$ is an admissible solution of the equation (1.1).

2. Proofs of Theorems 1.1 and 1.2. *Proof of Theorem* 1.1. Since $A(z)$ is an entire function, each pole of a meromorphic solution $\varphi(z)$ is a simple pole with residue 1 or 2 (see [4, pp. 321–322]). Hence there exists an entire function $f(z)$ such that $\varphi(z) = f'(z)/f(z)$. By simple computation, we see that $f(z)$ satisfies the linear differential equation of third order

(2.1) $$w''' + 4A(z)w' + 2A'(z)w = 0.$$

We know that a fundamental set of the equation (2.1) is given by $\{w_1^2, w_1w_2, w_1^2\}$, where $w_1(z)$, $w_2(z)$ are linearly independent solutions of linear differential equation of second order

(2.2) $$w'' + A(z)w = 0$$

(see e.g., [2, 2-8]). Thus we can write $f(z)$ as

$$f(z) = C_1w_1(z)^2 + C_2w_1(z)w_2(z) + C_3w_2(z)^2$$
$$= (c_1w_1(z) + c_2w_2(z))(c_3w_1(z) + c_4w_2(z)).$$

Put $\tilde{w}_1(z) := c_1w_1(z) + c_2w_2(z)$, $\tilde{w}_2(z) := c_3w_1(z) + c_4w_2(z)$. Then $\tilde{w}_1(z)$, $\tilde{w}_2(z)$ are also solutions of the equation (2.2). Define $u_1(z) := \tilde{w}_1'(z)/\tilde{w}_1(z)$, $u_2(z) := \tilde{w}_2'(z)/\tilde{w}_2(z)$. Then $u_1(z)$, $u_2(z)$ satisfy the Riccati equation (1.1). We immediately obtain $\varphi(z) = u_1(z) + u_2(z)$.

The existence of a meromorphic solution $\varphi(z)$ follows from the arguments above and from the existence theorem to the equation (2.2) with an entire coefficient $A(z)$.

Proof of Theorem 1.2. Define $f(z) := U_1(z, u_2(z))$. Then we have $U_2(z, u_2(z)) = f'(z) + f(z)u_2(z)$. From (1.3),

(2.3) $$\Phi(z, \varphi(z)) = 3u_1(z)f(z) + f'(z) + f(z)u_2(z) = 0.$$

Suppose that $f(z) \not\equiv 0$ in (2.3). Then we may write (2.3) as

(2.4) $$3u_1(z) + u_2(z) + \frac{f'(z)}{f(z)} = 0.$$

In this proof, for a transcendental meromorphic function $g(z)$, we call z_0 an admissible pole of $g(z)$ if z_0 is a pole of $g(z)$ and neither a pole nor a zero of $A(z)$. It is easy to see that the admissible solution $u_1(z)$ of the Riccati equation (1.1) possesses an admissible pole with residue 1. Let z_0 be an admissible pole of $u_1(z)$. We have that if z_0 is a pole of $f(z)$, then z_0 is a pole of $u_2(z)$. Then from (2.4), we see that either z_0 is a pole of $u_2(z)$, or z_0 is not a pole of $u_2(z)$ but a zero of $f(z)$. First we treat the case when z_0 is not a pole of $u_2(z)$ but a zero of $f(z)$. It is easy to see that the residue of the Laurent expansion of $f'(z)/f(z)$ at z_0 is a positive integer. This contradicts (2.4). Secondly we consider the case when z_0 is a pole of $u_2(z)$. It folllows from (2.4) that z_0 is a simple pole of $u_2(z)$. We denote by R the residue in the Laurent expansion of $u_2(z)$ at z_0. Write $f(z)$ in a neighbourhood of z_0 as

$$f(z) = C(z - z_0)^\nu + O(z - z_0)^{\nu+1}, \quad \text{as } z \to z_0, \quad C \neq 0, \quad \nu \geqq -2.$$

By the definition of $f(z)$, we see that $\nu \geqq -1$ if and only if $R = 1$. Using (2.4), we get

(2.5) $$3 + R + \nu = 0.$$

Hence if $R = 1$, then from (2.5), $4 = -\nu \leqq 1$, which is absurd. Hence, we have $R \neq 1$, which implies that $\nu = -2$. From (2.5), we get $R = -1$. We have

(2.6) $$N(r, u_1) \leqq N(r, u_2) + S(r, u_1).$$

Since $u_1(z)$ is an admissible solution of the Riccati equation (1.1), we have $m(r, u_1) = S(r, u_1)$. From (2.6),

(2.7) $$T(r, u_1) \leqq N(r, u_2) + S(r, u_1) \leqq T(r, u_2) + S(r, u_1).$$

It follows from (2.7) that a real function $\psi(r)$ that satisfies $\psi(r) = S(r, u_1)$ also satisfies $\psi(r) = S(r, u_2)$. Conversely, we assert that

(2.8) $$T(r, u_2) \leqq T(r, u_1) + S(r, u_2).$$

In fact, let z_1 be an admissible pole of $u_2(z)$. Then by our assumption, z_1 is a pole of $u_1(z)$ and a pole of $\varphi(z)$ simultaneously. Thus we have

(2.9) $$N(r, u_2) \leqq N(r, u_1) + S(r, u_2).$$

By means of the theorem on the logarithmic derivative, we have $m(r, f'/f) = S(r, f)$. Recalling $U_1(z, u_2)$ is a differential polynomial in u_2, for a real function $\psi(r)$, $\psi(r) = S(r, f)$ implies $\psi(r) = S(r, u_2)$. Hence from (2.4),

(2.10) $$m(r, u_2) \leqq m(r, u_1) + m\left(r, \frac{f'}{f}\right) = S(r, u_1) + S(r, u_2) = S(r, u_2).$$

Therefore, the assertion (2.8) follows from (2.9) and (2.10). Hence in the sequel we may write $S(r, u_1) = S(r, u_2)$ and we get

(2.11) $$T(r, u_1) = T(r, u_2) + S(r, u_2).$$

As seen in the arguments above, almost all poles of $u_2(z)$ are simple poles with residue -1. Write $u_2(z)$ in a neighbourhood of such z_1 as

(2.12) $$u_2(z) = \frac{-1}{z - z_1} + O(z - z_1), \quad \text{as } z \to z_1,$$

and we have

(2.13) $$\frac{f'(z)}{f(z)} = \frac{-2}{z - z_1} + O(z - z_1), \quad \text{as } z \to z_1,$$

in a neighbourhood of z_0. We define the counting function concerning common zeros of two meromorphic functions $f(z)$ and $g(z)$. Let $n(r, 0; f)_g$ be the number of common zeros of $f(z)$ and $g(z)$ in $|z| \leqq r$, each counted according to the multiplicity of the zero of $f(z)$. The counting function $N(r, 0, f)_g$ is defined in the usual way. The integrated counting function $\bar{N}(r, 0; f)_g (= \bar{N}(r, 0; g)_f)$ counts distinct common zeros of $f(z)$ and $g(z)$. We also see from the arguments above that $N(r, f'/f)_f := N(r, 0; f/f')_f = S(r, u_2)$. Define

(2.14) $$\sigma(z) := 2u_2(z) - \frac{f'(z)}{f(z)}.$$

Then from (2.12) and (2.13), z_1 is a regular point of $\sigma(z)$. This implies that $N(r, \sigma) = S(r, u_2)$. From (2.10) and (2.14), we get $m(r, \sigma) = S(r, u_2)$. Hence $\sigma(z)$ is a small function with respect to $u_2(z)$. Combining (2.4) and (2.14), we obtain $\varphi(z) = (1/3)\sigma(z)$. We see from our assumption and (2.11) that it is not possible for $\varphi(z)$ to be a small function with respect to $u_2(z)$. Therefore, we conclude that $f(z) \equiv 0$ otherwise $\varphi(z)$ is a small function with respect to $u_2(z)$. This means that $u_2(z)$ satisfies the Riccati equation

(1.1).

We can find the existence theorem to meromorphic solutions of the equation (1.1) and the study of the equations (1.2) and (2.1) in, for instance, [1] [5] [6]. Finally, we give a summarizing diagram below.

$$\begin{array}{ccc} w'' + A(z)w = 0 & \xrightarrow{f=w_1w_2} & f''' + 4A(z)f' + 2A'(z)f = 0 \\ \downarrow u = w'/w & & \downarrow \varphi = f'/f \\ u' + u^2 + A(z) = 0 & \xrightarrow{\varphi=u_1+u_2} & \varphi'' + 3\varphi'\varphi + \varphi^3 + 4A(z)\varphi + 2A'(z) = 0. \end{array}$$

References

[1] S. B. Bank, G. G. Gundersen, and I. Laine: Meromorphic solutions of the Riccati differential equation. Ann. Acad. Sci. Fenn. Ser. A I Math., **6**, 369–398 (1981).

[2] M. Greguš: Third Order Linear Differential Equations. D. Reide, Dordrecht, Boston, Lancaster, Tokyo (1987).

[3] W. K. Hayman: Meromorphic Functions. Clarendon Press, Oxford (1964).

[4] K. Ishizaki: Admissible solutions of second order differential equations. Tôhoku Math. Jour. Math., **44**, 305–325 (1992).

[5] ——: On the complex oscillation of linear differential equations of third order. Complex Variables Theory Appl. (to appear).

[6] I. Laine: Nevanlinna Theory and Complex Differential Equations. W. Gruyter, Berlin, New York (1992).

[7] R. Nevanlinna: Analytic Functions. Springer-Verlag, Berlin, Heidelberg, New York (1970).