35. *A Divisor Problem. I*

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§1. Introduction. Let α be a real number ≥ 1. For any integer $n \geq 1$, let $\tau_\alpha(n)$ be the number of the divisors d of n of the form $d = [\alpha m]$, where m is an integer and $[\cdot]$ is the Gauss symbol. We are concerned with the asymptotic behavior of the sum

$$\sum_{n\leq X} \tau_\alpha(n)$$

as $X \to \infty$.

When $\alpha = 1$, $\tau_\alpha(n)$ is the usual divisor function $\tau(n)$ and if we put

$$\sum_{n\leq X} \tau(n) = X\log X + (2\gamma - 1)X + \Delta(X)$$

with the Euler constant γ, then Dirichlet proved first that

$$\Delta(X) \ll \sqrt{X}$$

and Voronoi proved later that

$$\Delta(X) \ll X^{\frac{1}{3}}.$$

The refinement of these results, namely, Dirichlet's divisor problem, has been the subject of many mathematicians (cf. Chap. XII of Titchmarsh [6] and Iwaniec and Mozzochi [4], for example). In this article, we shall evaluate our sum for an irrational α. The sum for a rational α will be treated in the subsequent paper.

To state our result, let $\phi(x)$ be a non-decreasing positive function of $x \geq 1$. An irrational number α is said to be of type $< \phi$ if

$$q\,\|q\alpha\| \geq \frac{1}{\phi(q)} \text{ for all integer } q \geq 1,$$

where $\|x\| = \min(\{x\}, 1 - \{x\})$ and $\{x\}$ is the fractional part of x (cf. Kuipers and Niederreiter [5]). Now our result may be stated as follows.

Theorem. *Let α be an irrational number > 1 and $\frac{1}{\alpha}$ be of type $< \phi$. Then we have for $X > X_0$,*

$$\sum_{n\leq X} \tau_\alpha(n) = \frac{1}{\alpha}(X\log X + (2\gamma - 1)X) + X\left(\left\{\frac{1}{\alpha}\right\} - \sum_{n=1}^{\infty}\frac{\left\{\frac{n+1}{\alpha}\right\}}{n(n+1)}\right) + O(X^{\frac{2}{5}}\log(X\phi(X))).$$

Remark 1. To get the remainder term $O(\sqrt{X})$ for any α is simple if we estimate S_4 and S_5 below trivially. So to refine $O(\sqrt{X})$ up to the above remainder term will be the main part of this article.

Remark 2. It is more suggestive to write

$$X\left(\left\{\frac{1}{\alpha}\right\} - \sum_{n=1}^{\infty}\frac{\left\{\frac{n+1}{\alpha}\right\}}{n(n+1)}\right)$$

as

$$X\tilde{Z}_\alpha(1),$$

where we put for any real a and b and for $\Re s > 1$,

$$Z_{a,b}(s) = \sum_{n=1}^{\infty} \frac{\{an+b\} - \frac{1}{2}}{n^s},$$

$$Z_{a,0}(s) = Z_a(s)$$

and

$$\tilde{Z}_\alpha(s) = Z_{\frac{1}{\alpha}}(s) - Z_{\frac{1}{\alpha},\frac{1}{\alpha}}(s).$$

Because the generating Dirichlet series for $\tau_\alpha(n)$ is

$$\sum_{n=1}^{\infty} \frac{\tau_\alpha(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{[\alpha m]^s}\zeta(s) = \left(\frac{1}{\alpha}\zeta(s) + \tilde{Z}_\alpha(s)\right)\zeta(s)$$

and the function $\frac{X^s}{s}\sum_{n=1}^{\infty}\frac{\tau_\alpha(n)}{n^s}$ has a double pole at $s=1$ with the residue

$$\frac{1}{\alpha}(X\log X + (2\gamma-1)X) + X\tilde{Z}_\alpha(1),$$

where

$$\tilde{Z}_\alpha(1) = Z_{\frac{1}{\alpha}}(1) - Z_{\frac{1}{\alpha},\frac{1}{\alpha}}(1).$$

§2. Proof of theorem. We may suppose that X is an integer N. We put

$$\eta(d) = \begin{cases} 1 & \text{if } d \text{ is of the form } [\alpha m] \text{ with a positive integer } m \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha > 1$, we have $\eta(d) = \left[\frac{d+1}{\alpha}\right] - \left[\frac{d}{\alpha}\right]$.

Now

$$S \equiv \sum_{n\le N}\tau_\alpha(n) = \sum_{n\le N}\sum_{d|n}\eta(d)$$

$$= \sum_{d\le\sqrt{N}}\eta(d)\sum_{d|n,n\le N}\cdot 1 + \sum_{\sqrt{N}<d\le N}\eta(d)\sum_{d|n,n\le N}\cdot 1 = S_1 + S_2, \text{ say.}$$

$$S_1 = \sum_{d\le\sqrt{N}}\left[\frac{N}{d}\right]\eta(d) = \sum_{d\le\sqrt{N}}\left[\frac{N}{d}\right]\left(\left[\frac{d+1}{\alpha}\right] - \left[\frac{d}{\alpha}\right]\right)$$

$$= \frac{1}{\alpha}\sum_{d\le\sqrt{N}}\left[\frac{N}{d}\right] + \sum_{d\le\sqrt{N}}\left[\frac{N}{d}\right]\left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right) = \frac{1}{2\alpha}\sum_{n\le N}\tau(n) + \frac{1}{2\alpha}[\sqrt{N}]^2$$

$$+ N\sum_{d\le\sqrt{N}}\frac{1}{d}\left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right) - \sum_{d\le\sqrt{N}}\left\{\frac{N}{d}\right\}\left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right)$$

$$= \frac{1}{2\alpha}\sum_{n\le N}\tau(n) + \frac{1}{2\alpha}[\sqrt{N}]^2 + S_3 - S_4, \text{ say.}$$

$$S_3 = \sqrt{N}\sum_{1\le d\le\sqrt{N}}\left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right) + N\int_1^{\sqrt{N}}\sum_{1\le d\le t}\left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right)\frac{1}{t^2}dt$$

$$= \sqrt{N}\left\{\frac{1}{\alpha}\right\} - \sqrt{N}\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\} - N\int_1^{\sqrt{N}}\left(-\left\{\frac{1}{\alpha}\right\} + \left\{\frac{[t]+1}{\alpha}\right\}\right)\frac{1}{t^2}dt.$$

We shall estimate S_4 later and turn to estimate S_2.

$$S_2 = \sum_{\sqrt{N}<d\le N} \eta(d) \sum_{k\le \frac{N}{d}} .1 = \sum_{k\le\sqrt{N}} \sum_{\sqrt{N}<d\le\frac{N}{k}} \eta(d)$$
$$= \sum_{k\le\sqrt{N}} \sum_{\sqrt{N}<d\le\frac{N}{k}} \left(\left[\frac{d+1}{\alpha}\right] - \left[\frac{d}{\alpha}\right]\right)$$
$$= \frac{1}{\alpha} \sum_{k\le\sqrt{N}} \sum_{\sqrt{N}<d\le\frac{N}{k}} \cdot 1 + \sum_{k\le\sqrt{N}} \sum_{\sqrt{N}<d\le\frac{N}{k}} \left(\left\{\frac{d}{\alpha}\right\} - \left\{\frac{d+1}{\alpha}\right\}\right)$$
$$= \frac{1}{2\alpha} \sum_{n\le N} \tau(n) - \frac{1}{2\alpha}([\sqrt{N}])^2 + [\sqrt{N}]\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\} - \sum_{k\le\sqrt{N}} \left\{\frac{\left[\frac{N}{k}\right]+1}{\alpha}\right\}$$
$$= \frac{1}{2\alpha} \sum_{n\le N} \tau(n) - \frac{1}{2\alpha}([\sqrt{N}])^2 + [\sqrt{N}]\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\} - S_5, \text{ say.}$$

Consequently, we get

$$S = \frac{1}{\alpha} \sum_{n\le N} \tau(n) - N\int_1^{\sqrt{N}} \left(-\left\{\frac{1}{\alpha}\right\} + \left\{\frac{[t]+1}{\alpha}\right\}\right)\frac{1}{t^2}\,dt$$
$$- \{\sqrt{N}\}\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\} + \sqrt{N}\left\{\frac{1}{\alpha}\right\} - S_4 - S_5.$$

We shall estimate S_4 and S_5 in the next section.

§3. Completion of the proof. We shall evaluate S_5 first. Let δ satisfy $0 < \delta < \frac{1}{6}$

$$S_5 = \sum_{N^{\frac{1}{3}+\delta}\le k\le\sqrt{N}} \left\{\frac{\left[\frac{N}{k}\right]+1}{\alpha}\right\} + O(N^{\frac{1}{3}+\delta})$$
$$= \sum_{\substack{\sqrt{N}-1<n\le N^{\frac{2}{3}-\delta}\\ n(n+1)>N}} \left\{\frac{n+1}{\alpha}\right\}\left(\left[\frac{N}{n}\right] - \left[\frac{N}{n+1}\right]\right)$$
$$+ \sum_{\substack{\sqrt{N}-1<n\le N^{\frac{2}{3}-\delta}\\ n(n+1)\le N}} \left\{\frac{n+1}{\alpha}\right\} \sum_{\substack{N^{\frac{1}{3}+\delta}\le k\le\sqrt{N}\\ \left[\frac{N}{k}\right]=n}} \cdot 1 + O(N^{\frac{1}{3}+\delta})$$
$$= \left\{\frac{[\sqrt{N}]+1}{\alpha}\right\}\left[\frac{N}{[\sqrt{N}]}\right] + \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \left[\frac{N}{d}\right]\left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right) + O(N^{\frac{1}{3}+\delta})$$
$$= \left\{\frac{[\sqrt{N}]+1}{\alpha}\right\}\left[\frac{N}{[\sqrt{N}]}\right]$$
$$+ N \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \frac{1}{d}\left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)$$
$$- \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \left(\left\{\frac{N}{d}\right\} - \frac{1}{2}\right)\left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)$$
$$- \frac{1}{2} \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right) + O(N^{\frac{1}{3}+\delta})$$
$$= \left\{\frac{[\sqrt{N}]+1}{\alpha}\right\}\left[\frac{N}{[\sqrt{N}]}\right] + S_6 + S_7 + S_8 + O(N^{\frac{1}{3}+\delta}), \text{ say.}$$

$$S_6 = N^{\frac{1}{3}+\delta} \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)$$

$$+ N\int_{[\sqrt{N}]+1}^{N^{\frac{2}{3}-\delta}} \frac{1}{t^2} \sum_{[\sqrt{N}]+1\le d\le t} \left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)dt$$

$$= - N\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\}\int_{[\sqrt{N}]+1}^{N^{\frac{2}{3}-\delta}} \frac{1}{t^2}dt + N\int_{[\sqrt{N}]+1}^{N^{\frac{2}{3}-\delta}} \frac{1}{t^2}\left\{\frac{[t]+1}{\alpha}\right\}dt + O(N^{\frac{1}{3}+\delta})$$

$$= -\frac{N}{[\sqrt{N}]+1}\left\{\frac{[\sqrt{N}]+1}{\alpha}\right\} + N\int_{[\sqrt{N}]+1}^{\infty} \frac{1}{t^2}\left\{\frac{[t]+1}{\alpha}\right\}dt + O(N^{\frac{1}{3}+\delta}).$$

$$S_8 = O(1).$$

To estimate S_7, we notice that when $d \nmid N$,

$$\left\{\frac{N}{d}\right\} - \frac{1}{2} = - \sum_{1\le k\le\left[\frac{d}{N^\theta}\right]} \frac{1}{k\pi}\sin\left(2\pi k\frac{N}{d}\right) + O\left(\frac{N^\theta}{|R_N(d)|}\right),$$

where $0 < \theta < \frac{1}{2}$ and $R_N(d)$ is the least residue of N mod d (cf. p. 38 of Vinogradov [7]). Then

$$S_7 = \sum_{\substack{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta} \\ d\nmid N}} \sum_{1\le k\le\left[\frac{d}{N^\theta}\right]} \frac{1}{k\pi}\sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)$$

$$+ O\left(N^\theta \sum_{\substack{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta} \\ d\nmid N}} \frac{1}{|R_N(d)|}\right) + O(\tau(N))$$

$$= S_9 + O(S_{10}) + O(\tau(N)).$$

$$S_{10} \ll N^\theta \sum_{1\le j\le N-1} \frac{\tau(N+j)+\tau(N-j)}{j} \ll N^{\theta+\varepsilon}.$$

$$S_9 \ll \sum_{1\le k\ll N^{\frac{2}{3}-\delta-\theta}} \frac{1}{k}\left|\sum_{\substack{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta} \\ k\le\left[\frac{d}{N^\theta}\right]}} \sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d+1}{\alpha}\right\} - \left\{\frac{d}{\alpha}\right\}\right)\right| + \tau(N)\log N$$

$$\ll \sum_{1\le k\le\frac{[\sqrt{N}]+1}{N^\theta}} \frac{1}{k}\left|\sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d+1}{\alpha}\right\} - \frac{1}{2}\right)\right|$$

$$+ \sum_{1\le k\le\frac{[\sqrt{N}]+1}{N^\theta}} \frac{1}{k}\left|\sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d}{\alpha}\right\} - \frac{1}{2}\right)\right|$$

$$+ \sum_{\frac{[\sqrt{N}]+1}{N^\theta}\le k\le N^{\frac{2}{3}-\delta-\theta}} \frac{1}{k}\left|\sum_{kN^\theta\le d\le N^{\frac{2}{3}-\delta}} \sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d+1}{\alpha}\right\} - \frac{1}{2}\right)\right|$$

$$+ \sum_{\frac{[\sqrt{N}]+1}{N^\theta}\le k\le N^{\frac{2}{3}-\delta-\theta}} \frac{1}{k}\left|\sum_{kN^\theta\le d\le N^{\frac{2}{3}-\delta}} \sin\left(2\pi k\frac{N}{d}\right)\left(\left\{\frac{d}{\alpha}\right\} - \frac{1}{2}\right)\right| + \tau(N)\log N$$

$$= S_{11} + S_{12} + S_{13} + S_{14} + \tau(N)\log N, \text{ say.}$$

$$S_{11} \ll \sum_{1\le k\ll N^{\frac{1}{2}-\theta}} \frac{1}{k}\left|\sum_{1\le h\le H} \frac{1}{h} \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \sin\left(2\pi k\frac{N}{d}\right)\sin\left(2\pi h\frac{d+1}{\alpha}\right)\right|$$

$$+ \sum_{1\le k\ll N^{\frac{1}{2}-\theta}} \frac{1}{k} \sum_{[\sqrt{N}]+1\le d\le N^{\frac{2}{3}-\delta}} \frac{1}{H\left\|\frac{d+1}{\alpha}\right\|}$$

$$\ll \sum_{1\le k\ll N^{\frac12-\theta}} \frac{1}{k} \sum_{1\le h\le H} \frac{1}{h} \Big| \sum_{[\sqrt{N}]+1\le d\le N^{\frac23-\delta}} e\Big(k\frac{N}{d}+h\frac{d}{\alpha}\Big)\Big|$$
$$+ \sum_{1\le k\ll N^{\frac12-\theta}} \frac{1}{k} \sum_{1\le h\le H} \frac{1}{h} \Big| \sum_{[\sqrt{N}]+1\le d\le N^{\frac23-\delta}} e\Big(k\frac{N}{d}-h\frac{d}{\alpha}\Big)\Big|$$
$$+ \frac{N^{\frac23-\delta}\log N}{H}(\log N+\psi(CN^{\frac23-\delta})),$$

where we have used p. 131 of Kuipers and Niederreiter [5]. By Theorem 5.9 of Titchmarsh [6], we get for $D\le D'\le 2D$

$$\sum_{D\le d\le D'} e\Big(k\frac{N}{d}\pm h\frac{d}{\alpha}\Big) \ll D\sqrt{\frac{kN}{D^3}}+\sqrt{\frac{D^3}{kN}}.$$

Applying this to the last two sums and choosing $H=N^{\frac23-\delta}\psi(CN^{\frac23-\delta})$, we get

$$S_{11} \ll \sum_{1\le k\ll N^{\frac12-\theta}} \frac{1}{k} \sum_{1\le h\le H} \frac{1}{h}\Big(\frac{\sqrt{kN}}{N^{\frac14}}+\frac{N^{1-\frac32\delta}}{\sqrt{kN}}\Big)$$
$$+ \frac{N^{\frac23-\delta}\log N}{H}(\log N+\psi(CN^{\frac23-\delta}))$$
$$\ll (N^{\frac12-\frac12\theta}+N^{\frac12-\frac32\delta})\log(N\psi(N)).$$

In a similar manner, we get

$$S_{12}+S_{13}+S_{14} \ll (N^{\frac12-\frac12\theta}\log N+N^{\frac12-\frac32\delta})\log(N\psi(N)).$$

By choosing $\delta=\frac{1}{15}$ and $\theta=\frac13$, we get

$$S_5=\Big\{\frac{[\sqrt{N}]+1}{\alpha}\Big\}\Big[\frac{N}{[\sqrt{N}]}\Big]-\frac{N}{[\sqrt{N}]+1}\Big\{\frac{[\sqrt{N}]+1}{\alpha}\Big\}$$
$$+N\int_{[\sqrt{N}]+1}^{\infty}\frac{1}{t^2}\Big\{\frac{[t]+1}{\alpha}\Big\}dt+O(N^{\frac25}\log(N\psi(N))).$$

Finally, we shall evaluate S_4.

$$S_4=\sum_{d\le\sqrt{N}}\Big(\Big\{\frac{N}{d}\Big\}-\frac12\Big)\Big(\Big\{\frac{d}{\alpha}\Big\}-\frac12\Big)$$
$$-\sum_{d\le\sqrt{N}}\Big(\Big\{\frac{N}{d}\Big\}-\frac12\Big)\Big(\Big\{\frac{d+1}{\alpha}\Big\}-\frac12\Big)+O(1)$$
$$=S_{15}+S_{16}+O(1),\text{ say.}$$

As above, we get

$$S_{15} = -\sum_{N^{\frac13}\le d\le\sqrt{N}}\ \sum_{1\le k\le\left[\frac{d}{N^{\frac13}}\right]}\frac{1}{k\pi}\sin\Big(2\pi k\frac{N}{d}\Big)\Big(\Big\{\frac{d}{\alpha}\Big\}-\frac12\Big)$$
$$+O\Big(N^{\frac13}\sum_{N^{\frac13}\le d\le\sqrt{N}}\frac{1}{|R_N(d)|}\Big)+O(N^{\frac13})$$
$$\ll \sum_{1\le k\ll N^{\frac16}}\frac{1}{k}\Big|\sum_{\substack{N^{\frac13}\le d\le\sqrt{N}\\ k\le\left[\frac{d}{N^{\frac13}}\right]}}\sin\Big(2k\pi\frac{N}{d}\Big)\Big(\Big\{\frac{d}{\alpha}\Big\}-\frac12\Big)\Big|+N^{\frac13+\varepsilon}$$

$$\ll \sum_{1\le k\ll N^{\frac{1}{6}}} \frac{1}{k} \sum_{1\le h\le H'} \frac{1}{h} \left(\frac{\sqrt{kN}}{\sqrt{kN^{\frac{1}{3}}}} + \frac{N^{\frac{3}{4}}}{\sqrt{kN}} \right)$$

$$+ \frac{\sqrt{N}\log N}{H'} (\log N + \phi(C\sqrt{N}))$$

$$\ll N^{\frac{1}{3}} \log N \log(N\phi(N)),$$

where we take $H' = \sqrt{N}\, \phi(C\sqrt{N})$.

Treating S_{16} similarly, we get

$$S_4 = O(N^{\frac{1}{3}+\varepsilon}) + O\,(N^{\frac{1}{3}} \log N \log(N\phi(N))).$$

Thus we get

$$S = \frac{1}{\alpha} \sum_{n\le N} \tau(n) - N\int_1^{\sqrt{N}} \left(-\left\{\frac{1}{\alpha}\right\} + \left\{\frac{[t]+1}{\alpha}\right\} \right) \frac{1}{t^2}\, dt + \sqrt{N} \left\{\frac{1}{\alpha}\right\}$$

$$- N\int_{[\sqrt{N}]+1}^{\infty} \frac{1}{t^2} \left\{\frac{[t]+1}{\alpha}\right\} dt + O(N^{\frac{2}{5}} \log(N\phi(N))$$

$$= \frac{1}{\alpha} \sum_{n\le N} \tau(n) - N\int_1^{\infty} \left\{\frac{[t]+1}{\alpha}\right\} \frac{1}{t^2}\, dt + \left\{\frac{1}{\alpha}\right\} N$$

$$+ O(N^{\frac{2}{5}} \log(N\phi(N)))$$

$$= \frac{1}{\alpha} \sum_{n\le N} \tau(n) - N \sum_{n=1}^{\infty} \frac{\left\{\frac{n+1}{\alpha}\right\}}{n(n+1)} + \left\{\frac{1}{\alpha}\right\} N$$

$$+ O(N^{\frac{2}{5}} \log(N\phi(N))).$$

This completes our proof of Theorem.

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