29. *Eigenvalues of the Laplace-Beltrami Operator and the von-Mangoldt Function*

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§ 1. **Introduction.** Let $\Lambda(n)$ be the von-Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime number } p \text{ and an integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator on $L^2(H/\Gamma)$, where H is the upper half of the complex plane and we take $\Gamma = PSL(2, Z)$. It is well known that $\lambda_1 > \frac{1}{4}$. We put $\lambda_j = \frac{1}{4} + r_j^2$ for $j \geq 0$. Here we are concerned with the relation between $\Lambda(n)$ and λ_j.

We recall first the following result which has been proved in the author's [1]-[3]. It shows that an individual $\Lambda(n)$ can be expressed by the eigen-values. For any positive α, let $Z_\alpha(s)$ be defined by

$$Z_\alpha(s) = \sum_{r_j>0} \frac{\sin(\alpha r_j)}{r_j^s} \quad \text{for} \quad \Re(s) > 1.$$

This function is shown to be an entire function of s. Then we have

(I) : *For any integer* $n \geq 2$,

$$\Lambda(n) = \pi n \lim_{\alpha \to 2\log n} (\alpha - 2\log n) Z_\alpha(0).$$

We recall secondly the following result which has been proved by Venkov [11]. It shows that the sum

$$\sum_{n \leq X} \Lambda(n)$$

can be written down in terms of the eigen-values and the remainder term of the prime geodesic theorem. Let $\{P\}$ run over the set of the hyperbolic conjugacy classes of the conjugacy elements in Γ and $N(P) = N(P_0)^k$, where $\{P\} = \{P_0\}^k$ with the primitive hyperbolic conjugacy class $\{P_0\}$ and $N(P_0)$ is the norm of P_0. We put

$$\tilde{\Lambda}(P) = \frac{\log N(P_0)}{1 - (N(P))^{-1}}$$

and

$$\tilde{\psi}(X) = \sum_{N(P) \leq X} \tilde{\Lambda}(P).$$

Then Venkov's result may be stated as follows.

(II): *For* $X > X_0$, *we have*

$$\sum_{n \leq X} \Lambda(n) = \frac{X}{2} \lim_{t \to +0} \sum_{r_j>0} \frac{\cos(2r_j \log X) + 2r_j \sin(2r_j \log X)}{\lambda_j} e^{-t\lambda_j}$$
$$- \frac{1}{2}(\tilde{\psi}(X^2) - X^2) + O(X^\varepsilon),$$

where ε *is an arbitrary positive number.*

Here we should recall the prime geodesic theorem (cf. Huber [5] and Iwaniec [7]) which states that for any positive ε and for $X > X_o$,

$$\tilde{\psi}(X) - X = O(X^{\frac{35}{48}+\varepsilon})$$

and

$$\textstyle\sum_{N(P_o)\leq X} \log N(P_o) - X = O(X^{\frac{35}{48}+\varepsilon}),$$

where $\{P_0\}$ runs over the set of all primitive hyperbolic conjugacy classes of the conjugacy elements in Γ.

In this article, we shall supplement to these by adding another explicit relation between $\Lambda(n)$ and λ_j. It shows that the average of the sum of $\Lambda(n)$, namely,

$$\int_0^X \sum_{n\leq y} \Lambda(n)\,dy$$

can be written down in terms of the eigen-values and the average of the remainder term of the prime geodesic theorem. To state our result, let $Z(s)$ be the Selberg zeta function defined by

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - (N(P_0))^{-s-k}) \text{ for } \Re s > 1.$$

It is well-known that $Z(s)$ has meromorphic continuation to the whole complex plane (cf. Selberg [10]). $Z(s)$ satisfies the following functional equation (cf. Selberg [10], Venkov [12], Vigneras [13] and Kurokawa [8]).

$$\begin{aligned}\frac{Z'}{Z}(s) + \frac{Z'}{Z}(1-s) = &-\left(s-\frac{1}{2}\right)\frac{\pi}{3}\cot(\pi s) + 2\log\frac{2}{\pi} + 2\frac{\zeta'}{\zeta}(2-2s) + 2\frac{\zeta'}{\zeta}(2s)\\ &+\frac{\Gamma'}{\Gamma}(1-s) + \frac{\Gamma'}{\Gamma}(s) + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}-s\right) + \frac{\Gamma'}{\Gamma}\left(s+\frac{1}{2}\right)\\ &+\frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) + \frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - \frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{1+s}{2}\right) - \frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{2-s}{2}\right)\\ &+\frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{s}{3}\right) + \frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{3}\right) - \frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{s+2}{3}\right) - \frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{3-s}{3}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function and $\Gamma(s)$ is the Γ-function.

The position and the multiplicities of the zeros and poles of $Z(s)$ are well-known (cf. Venkov [12] and p. 141 of Kurokawa [9]). In particular, we put

$$A_o = \lim_{s\to-\frac{1}{2}}\left(\frac{Z'}{Z}(s) + \frac{1}{s+\frac{1}{2}}\right)$$

and

$$B_o = \lim_{s\to 0}\left(\frac{Z'}{Z}(s) + \frac{1}{s}\right).$$

From the above functional equation it is easily shown that

$$\begin{aligned}A_o = &-\frac{Z'}{Z}\left(\frac{3}{2}\right) + 2\left(\frac{\zeta'}{\zeta}(3) - \frac{\zeta'}{\zeta}(2)\right) + 2\log 2 + \frac{10}{3} + \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) + \frac{\Gamma'}{\Gamma}(2)\\ &+\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{5}{4}\right)\right) + \frac{2}{9}\left(\frac{\Gamma'}{\Gamma}\left(\frac{5}{6}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{7}{6}\right)\right)\end{aligned}$$

and

$$B_o = -\lim_{s\to 0}\left(\frac{Z'}{Z}(1-s) + \frac{1}{s}\right) + 2\frac{\Gamma'}{\Gamma}(1) + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)$$

$$+\frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{1}{3}\right)-\frac{2}{9}\frac{\Gamma'}{\Gamma}\left(\frac{2}{3}\right)+\frac{5}{3}+2\frac{\zeta'}{\zeta}(2)+4\log 2,$$

where we have

$$\lim_{s\to 0}\left(\frac{Z'}{Z}(1-s)+\frac{1}{s}\right)=1+\int_1^\infty \frac{\sum_{N(P)\le y}\tilde{\Lambda}(P)-y}{y^2}\,dy.$$

Now our result may be stated as follows.

Theorem. *For $X>1$, we have*

$$\int_0^X \sum_{n\le y}\Lambda(n)\,dy=\frac{X^2}{2}\Re\Big(\sum_{r_j>0}\frac{X^{2ir_j}}{\left(\frac{1}{2}+ir_j\right)(1+ir_j)}\Big)-\frac{1}{2}\int_0^X\Big(\sum_{N(P)\le y^2}\tilde{\Lambda}(P)-y^2\Big)\,dy$$
$$-X\log X+X\left(\frac{1}{2}B_o+1-\log 2\pi\right)+\log X$$
$$+\log 2\pi-\frac{1}{2}A_o+C_o-\frac{\zeta'}{\zeta}(2)+G(X),$$

where A_o and B_o are defined above, C_o is the Euler constant and we put

$$G(X)=\sum_{k=1}^{\infty}\frac{X^{-2k}}{k(2k+1)}-\sum_{k=1}^{\infty}\frac{X^{1-2k}}{k(2k-1)}$$
$$+\frac{1}{2}X^3\sum_{\substack{k=2\\ k\equiv 1(6)}}^{\infty}\frac{X^{-2k}}{(k-1)(2k-3)}\left(2\left[\frac{k}{6}\right]-1\right)$$
$$+\frac{1}{2}X^3\sum_{\substack{k=2\\ k\not\equiv 1(6)}}^{\infty}\frac{X^{-2k}}{(k-1)(2k-3)}\left(2\left[\frac{k}{6}\right]+1\right).$$

We shall prove our theorem in a standard way (cf. Ingham [6] and Hejhal [4]) and some of the details and references will be omitted.

§2. Proof of Theorem. Let N be a sufficiently large integer. We can choose $T=T_N$ in $N\le T\le N+1$ such that $Z(s)$ has no zeros or poles in $|\Im s-T|\le\frac{C}{T}$ with some positive constant C. Let $b=1+\delta$ with $\delta=\frac{1}{\log T}$. We shall evaluate the integral

$$I\equiv\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{Z'}{Z}(s)\frac{X^{2s+1}}{s(2s+1)}\,ds.$$

We have on one hand

$$I=X\sum_{\{P\}}\tilde{\Lambda}(P)\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{\left(\frac{X}{\sqrt{N(P)}}\right)^{2s}}{s(2s+1)}\,ds=X\sum_{N(P)\le X^2}\tilde{\Lambda}(P)\left(1-\frac{\sqrt{N(P)}}{X}\right)$$
$$+O\Big(\frac{X}{T}\sum_{\{P\}}\tilde{\Lambda}(P)\Big(\frac{X}{\sqrt{N(P)}}\Big)^{2b}\min\Big(1,\frac{1}{T\left|\log\frac{X}{\sqrt{N(P)}}\right|}\Big)\Big)=I_1+I_2,\text{ say.}$$

It is easily seen that

$$I_1=\int_0^X\sum_{N(P)\le y^2}\tilde{\Lambda}(P)\,dy.$$

$$I_2\ll\frac{X^{1+2b}}{T}\Big(\sum_{\substack{N(P)\le\frac{X^2}{2}\\ N(P)\ge 2X^2}}+\sum_{\frac{X^2}{2}\le N(P)\le X^2-h}+\sum_{X^2-h\le N(P)\le X^2+h}+\sum_{X^2+h\le N(P)\le 2X^2}\Big)$$

$$\cdot \frac{\tilde{\Lambda}(P)}{N(P)^{1+\delta}} \min\left(1, \frac{1}{T\left|\log \frac{X}{\sqrt{N(P)}}\right|}\right) = I_3 + I_4 + I_5 + I_6, \text{ say,}$$

where we take $h = X^{\frac{35}{24}+\varepsilon}$

$$I_4 \ll \frac{X^3}{T^2} \sum_{\frac{X^2}{2} \le N(P) \le X^2 - h} \frac{\tilde{\Lambda}(P)}{X^2 - N(P)}$$

$$\ll \frac{X^3}{T^2} \sum_{1 \le m \ll \frac{X^2}{h}} \frac{1}{mh} \left(\sum_{X^2-(m+1)h \le N(P_0) \le X^2 - mh} \log N(P_o) + \frac{h}{X} + X^{\frac{35}{48}+\varepsilon} \right)$$

$$\ll \frac{X^3}{T^2} \sum_{1 \le m \ll \frac{X^2}{h}} \frac{1}{mh} (h + X^{\frac{35}{24}+\varepsilon}) \ll \frac{X^3}{T^2} \log X.$$

Similarly, we get, using the prime geodesic theorem stated above,

$$I_2 \ll \frac{X^{3+2\delta}}{T^2} \log T + \frac{X^3}{T^2} \log X + \frac{X^{\frac{59}{24}+\varepsilon}}{T}.$$

Thus we get

$$I = \int_0^X \sum_{N(P) \le y^2} \tilde{\Lambda}(P)\, dy + O\left(\frac{X^{3+2\delta}}{T^2} \log T + \frac{X^3}{T^2} \log X + \frac{X^{\frac{59}{24}+\varepsilon}}{T}\right).$$

On the other hand, by Cauchy's theorem,

$$I = I_7 + \frac{X^3}{3} + \frac{1}{2} \sum_{|r_j| \le T, j \ge 1} \frac{X^{2+2ir_j}}{\left(\frac{1}{2} + ir_j\right)(1 + ir_j)} + 2 \sum_{|\gamma| \le 2T} \frac{X^{\rho+1}}{\rho(\rho+1)} - X^2$$

$$+ X^3 \sum_{\substack{2 \le k \le D+1 \\ k \equiv 1(6)}} \frac{X^{-2k}}{(1-k)(2-2k+1)} \left(2\left[\frac{k}{6}\right] - 1\right)$$

$$+ X^3 \sum_{\substack{2 \le k \le D+1 \\ k \not\equiv 1(6)}} \frac{X^{-2k}}{(1-k)(2-2k+1)} \left(2\left[\frac{k}{6}\right] + 1\right)$$

$$+ X^2 \sum_{2 \le l \le D+\frac{1}{2}} \frac{X^{-2l}}{\left(\frac{1}{2} - l\right)(2-2l)} + X(B_o + 2 - 2\log X)$$

$$- A_o + 2 + 2\log X,$$

where ρ runs over the non-trivial zeros of $\zeta(s)$, $\gamma = \Im\, \rho$, D is sufficiently large and not near any "integer" or "integer-$\frac{1}{2}$", p. 141 of Kurokawa [9] is used and we put

$$I_7 = -\frac{1}{2\pi i} \left(\int_{b+iT}^{-D+iT} + \int_{-D+iT}^{-D-iT} + \int_{-D-iT}^{b-iT} \right) \frac{Z'}{Z}(s) \frac{X^{2s+1}}{s(2s+1)} ds$$

$$= I_8 + I_9 + I_{10}, \text{ say.}$$

To estimate these terms, we shall use the functional equation of $Z(s)$ as stated in the introduction.

We get first

$$I_9 \ll X^{1-2D}.$$

We decompose I_8 further as follows.

$$I_8 = \frac{1}{2\pi i} \left(\int_{\frac{1}{2}+\delta+iT}^{b+iT} + \int_{\frac{1}{2}-\delta+iT}^{\frac{1}{2}+\delta+iT} + \int_{-1+iT}^{\frac{1}{2}-\delta+iT} + \int_{-D+iT}^{-1+iT} \right) \frac{Z'}{Z}(s) \frac{X^{2s+1}}{s(2s+1)} ds$$

$= I_{11} + I_{12} + I_{13} + I_{14}$, say.

We get first

$$I_{11} \ll \int_{\frac{1}{2}+\delta}^{b} T^{2\max(0,1-\sigma)} \log T \frac{X^{2\sigma+1}}{T^2}\, d\sigma \ll \frac{X^{3+2\delta} \log T}{T}.$$

To estimate I_{12}, we use the following expression of $\frac{Z'}{Z}(s)$:

$$\frac{Z'}{Z}(\sigma + iT) = \sum_{\tilde{\rho}}{}^{*} \frac{1}{\sigma + iT - \tilde{\rho}} + O(T),$$

where $\tilde{\rho}$ runs over all the zeros of $Z(s)$ in the critical strip such that

$$\left| \tilde{\rho} - \frac{6}{5} - iT \right| \leq 2.$$

Now

$$I_{12} = \frac{1}{2\pi i} \sum_{\tilde{\rho}}{}^{*} \int_{\frac{1}{2}-\delta+iT}^{\frac{1}{2}+\delta+iT} \frac{X^{2s+1}}{(s-\tilde{\rho})s(2s+1)}\, ds + O\Big(\frac{X^{2+2\delta}}{T\log T}\Big)$$

$$\ll \frac{X^{2+2\delta}}{T} \log T.$$

$$I_{13} = \frac{1}{2\pi i} \int_{-1+iT}^{\frac{1}{2}-\delta+iT} \Big(-\frac{Z'}{Z}(1-s) + O(T)\Big) \frac{X^{2s+1}}{s(2s+1)}\, ds$$

$$\ll \int_{\frac{1}{2}+\delta}^{1} T^{2(1-\sigma)} \log T \frac{X^{3-2\sigma}}{T^2}\, d\sigma + \frac{X^{2-2\delta}}{T} \ll \frac{X^{2-2\delta}}{T}.$$

Using the functional equation of $Z(s)$ as stated in the introduction, we get

$$I_{14} \ll \frac{1}{XT}.$$

Thus we get

$$S_8 = O\Big(\frac{X^{3+2\delta}}{T} \log T\Big).$$

Estimating S_{10} in the same manner, we get

$$I_7 = O(X^{1-2D}) + O\Big(\frac{X^{3+2\delta}}{T} \log T\Big).$$

Consequently, letting D tend to ∞, we get

$$I = \frac{X^3}{3} + \frac{1}{2} \sum_{|r_j|\leq T, j\geq 1} \frac{X^{2+2ir_j}}{\Big(\frac{1}{2}+ir_j\Big)(1+ir_j)} + 2 \sum_{|\gamma|\leq 2T} \frac{X^{\rho+1}}{\rho(\rho+1)} - X^2$$

$$+ X^3 \sum_{\substack{k=2 \\ k\equiv 1(6)}}^{\infty} \frac{X^{-2k}}{(1-k)(2-2k+1)} \Big(2\Big[\frac{k}{6}\Big] - 1\Big)$$

$$+ X^3 \sum_{\substack{k=2 \\ k\not\equiv 1(6)}}^{\infty} \frac{X^{-2k}}{(1-k)(2-2k+1)} \Big(2\Big[\frac{k}{6}\Big] + 1\Big)$$

$$+ X^2 \sum_{l=2}^{\infty} \frac{X^{-2l}}{\Big(\frac{1}{2}-l\Big)(2-2l)} + X(B_o + 2 - 2\log X)$$

$$- A_o + 2 + 2\log X + O\Big(\frac{X^{3+2\delta}}{T} \log T\Big).$$

Letting N tend to ∞ and combining two expressions of I, we get first the following theorem.

Theorem. *For $X > 1$, we have*

$$\sum_{\rho} \frac{X^{\rho+1}}{\rho(\rho+1)} = \frac{1}{2}\int_0^X \Big(\sum_{N(P)\le y^2} \tilde{\Lambda}(P) - y^2 \Big) dy - \frac{1}{2}\Re\Big(\sum_{r_j>0} \frac{X^{2+2ir_j}}{\left(\frac{1}{2}+ir_j\right)(1+ir_j)} \Big)$$
$$+ \frac{1}{2}X^2 - \frac{1}{2}X^3 \sum_{\substack{k=2 \\ k\equiv 1(6)}}^{\infty} \frac{X^{-2k}}{(1-k)(2-2k+1)} \left(2\left[\frac{k}{6}\right]-1\right)$$
$$- \frac{1}{2}X^3 \sum_{\substack{k=2 \\ k\not\equiv 1(6)}}^{\infty} \frac{X^{-2k}}{(1-k)(2-2k+1)} \left(2\left[\frac{k}{6}\right]+1\right)$$
$$- \frac{X^2}{4}\sum_{l=2}^{\infty} \frac{X^{-2l}}{\left(\frac{1}{2}-l\right)(1-l)} - \frac{1}{2}X(B_o + 2 - 2\log X)$$
$$- \frac{1}{2}(-A_o + 2 + 2\log X).$$

Since, by p. 73 of Ingham [6],

$$\sum_{\rho} \frac{X^{\rho+1}}{\rho(\rho+1)} = \frac{X^2}{2} - \int_0^X \sum_{n\le y} \Lambda(n)dy - X\frac{\zeta'}{\zeta}(0) + \frac{\zeta'}{\zeta}(-1) - \sum_{k=1}^{\infty} \frac{X^{1-2k}}{2k(2k-1)},$$

we get our theorem as stated in the introduction.

References

[1] A. Fujii: Zeros, eigenvalues and arithmetic. Proc. Japan Acad., **60A,** 22–25 (1984).

[2] ——: A zeta function connected with eigenvalues of the Laplace-Beltrami operator on the fundamental domain of the modular group. Nagoya J. of Math., **60,** 22–25 (1984).

[3] ——: Arithmetic of some zeta functions connected with the eigenvalues of the Laplace Beltrami operator. Advanced Studies in Pure Math., **13,** 237–260 (1988).

[4] D. Hejhal: The Selberg trace formula for *PSL*(2, *R*). 1 (2). Lect. Notes in Math., vol. 548 (1002), Berlin, New York, 1976 (1983).

[5] H. Huber: Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. II. Math. Ann., **142,** 463–464 (1961).

[6] A. E. Ingham: The Distribution of Prime Numbers. Cambridge Univ. Press (1990).

[7] H. Iwaniec: Prime geodesic theorem. Crelle J., **349,** 136–159 (1984).

[8] N. Kurokawa: Parabolic components of zeta functions. Proc. Japan Acad., **64A,** 21–24 (1988).

[9] ——: Special values of Selberg zeta function. Contemporary Math., **83,** 133–150 (1989).

[10] A. Selberg: Collected Papers. Springer-Verlag, vol. 1 (1989); vol. 2 (1991).

[11] A. B. Venkov: A formula for the Chebyshev ψ-function. Math. Notes, **23,** 271–274 (1978).

[12] ——: On the remainder term in the Weyl-Selberg asymptotic formula. J. of Soviet Math., **17,** 2083–2097 (1981).

[13] M. F. Vigneras: L'equation fonctionelle de la fonction zeta de Selberg du groupe modulaire *PSL*(2, *Z*). Astérique, **61,** 235–249 (1979).