17. *New Criteria for Multivalent Meromorphic Starlike Functions of Order Alpha*

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Abstract: Let $M_{n+p-1}(\alpha)$ $(p \in N=\{1, 2, \ldots\}, n > -p.\ 0 \leq \alpha < p)$ deonte the class of functions of the form

$$f(z)=\frac{1}{z^{p}}+\frac{a_{0}}{z^{p-1}}+\frac{a_{1}}{z^{p-2}}+\cdots$$

which are regular and p-valent in the punctured disc $U^{*}=\{z: 0<|z|<1\}$ and satisfy the condition

$$\operatorname{Re}\left\{\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}-(p+1)\right\}<-\frac{p(n+p-1)+\alpha}{n+p},\ |z|<1,$$

$0 \leq \alpha < p$, where

$$D^{n+p-1} f(z)=\frac{1}{z^{p}(1-z)^{n+p}} * f(z) \quad (n>-p).$$

It is proved that $M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$ $(0 \leq \alpha < p, n > -p)$. Since $M_{0}(\alpha)$ is the class of p-valent meromorphically starlike functions of order $\alpha (0 \leq \alpha < p)$, all functions in $M_{n+p-1}(\alpha)$ are p-valent meromorphically starlike functions of order α. Further we consider the integrals of functions in $M_{n+p-1}(\alpha)$.

1. Introduction. Let Σ_{p} denote the class of functions of the form

$$f(z)=\frac{1}{z^{p}}+\frac{a_{0}}{z^{p-1}}+\frac{a_{1}}{z^{p-2}}+\ldots \quad (p \in N=\{1,2 \ldots\}) \tag{1.1}$$

which are regular and p-valent in the punctured disc $U^{*}=\{z: 0<|z|<1\}$ and let n be any integer greater than $-p$. A function $f(z)$ in Σ_{p} is said to be p-valent meromorphically starlike of order $\alpha (0 \leq \alpha < p)$ if and only if

$$\operatorname{Re}\left\{\frac{z f^{\prime}(z)}{f(z)}\right\}<-\alpha \quad \text{for } |z|<1. \tag{1.2}$$

The Hadamard product or convolution of two functions f, g in Σ_{p} will be denoted by $f * g$. Let

$$D^{n+p-1} f(z)=\frac{1}{z^{p}(1-z)^{n+p}} * f(z) \quad (n>-p) \tag{1.3}$$

$$=\frac{1}{z^{p}}\left[\frac{z^{n+2 p-1} f(z)}{(n+p-1)!}\right]^{(n+p-1)} \tag{1.4}$$

$$=\frac{1}{z^{p}}+\frac{n+p}{z^{p-1}} a_{0}+\frac{(n+p)(n+p+1)}{2!\, z^{p-2}} a_{1}+\cdots. \tag{1.5}$$

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In this paper along with other things we shall show that a function $f(z) \in \sum_p$ which satisfies one of the conditions

$$\text{(1.6)} \quad \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (p+1)\right\} < -\frac{p(n+p-1)+\alpha}{n+p}, \quad |z|<1,$$

for some $\alpha(0 \leq \alpha < p)$ and $n \in N_0 = N \cup \{0\}$, is meromorphically p-valent starlike in U^*. More precisely, it is proved that, for the classes $M_{n+p-1}(\alpha)$ of functions in $\sum_p$ satisfying (1.6).

$$\text{(1.7)} \qquad M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \quad (0 \leq \alpha < p,\ n > -p)$$

holds. Since $M_0(\alpha)$ equals $\sum_p^*(\alpha)$ (the class of meromorphically p-valent starlike functions of order α [5]), it follows from (1.7) that all functions in $M_{n+p-1}(\alpha)$ are p-valent meromorphically starlike of order α. Further for $c > p-1$, let

$$\text{(1.8)} \qquad F(z) = \frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)\,dt,$$

it is shown that $F(z) \in M_{n+p-1}(\alpha)$ whenever $f(z) \in M_{n+p-1}(\alpha)$. Also it is shown that if $f(z) \in M_{n+p-1}(\alpha)$ then

$$\text{(1.9)} \qquad F(z) = \frac{n+p}{z^{n+2p}}\int_0^z t^{n+2p-1} f(t)\,dt$$

belongs to $M_{n+p}(\alpha)$. Some known results of Bajpai [1], Goel and Sohi [3], Ganigi and Uralegaddi [2] and Uralegaddi and Ganigi [7] are extended. In [6] Ruscheweyh obtained the new criteria for univalent functions.

2. The classes $M_{n+p-1}(\alpha)$. In proving our main results (Theorems 1 and 2 below). We shall need the following lemma due to I. S. Jack [4].

Lemma. *Let $w(z)$ be non-constant and regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = k\, w(z_0)$, where k ls a real number and $k \geq 1$.*

Theorem 1. *$M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$, $0 \leq \alpha < p$ and n is any integer greater than $-p$.*

Proof. Let $f(z) \in M_{n+p}(\alpha)$. Then

$$\text{(2.1)} \qquad \mathrm{Re}\left\{\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} - (p+1)\right\} < -\frac{p(n+p)+\alpha}{n+p}.$$

We have to show that (2.1) implies the inequality

$$\text{(2.2)} \qquad \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (p+1)\right\} < -\frac{p(n+p-1)+\alpha}{n+p}.$$

Define $w(z)$ in U by

$$\text{(2.3)} \quad \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (p+1) = -\left\{\frac{p(n+p-1)+\alpha}{n+p} + \frac{p-\alpha}{n+p}\,\frac{1-w(z)}{1+w(z)}\right\}.$$

Clearly $w(z)$ is regular and $w(0) = 0$. Equation (2.3) may be written as

$$\text{(2.4)} \qquad \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{(n+p) + (n+3p-2\alpha)w(z)}{(n+p)(1+w(z))}.$$

Differentiating (2.4) logarithmically and using the identity

$$\text{(2.5)} \qquad z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z),$$

we obtain

$$(2.6)\quad \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-(p+1)+\frac{p(n+p)+\alpha}{n+p+1}$$
$$=\frac{p-\alpha}{n+p+1}\left\{-\frac{1-w(z)}{1+w(z)}+\frac{2zw'(z)}{(1+w(z))[n+p+(n+3p-2\alpha)w(z)]}\right\}.$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$(2.7)\quad z_0w'(z_0)=k\,w(z_0),$$

where $|w(z_0)|=1$ and $k\geq 1$. From (2.6) and (2.7) we obtain

$$(2.8)\quad \frac{D^{n+p+1}f(z_0)}{D^{n+p}f(z_0)}-(p+1)+\frac{p(n+p)+\alpha}{n+p+1}$$
$$=\frac{p-\alpha}{n+p+1}\left\{-\frac{1-w(z_0)}{1+w(z_0)}+\frac{2k\,w(z_0)}{(1+w(z_0))[n+p+(n+3p-2\alpha)w(z_0)]}\right\}.$$

Thus

$$(2.9)\quad \operatorname{Re}\left\{\frac{D^{n+p+1}f(z_0)}{D^{n+p}f(z_0)}-(p+1)+\frac{p(n+p)+\alpha}{n+p+1}\right\}$$
$$\geq\frac{p-\alpha}{2(n+p+1)(n+2p-\alpha)}>0,$$

which contradicts (2.1). Hence $|w(z)|<1$ and from (2.3) it follows that $f(z)\in M_{n+p-1}(\alpha)$.

Theorem 2. *Let $f(z)\in\sum_p$ satisfy the condition*

$$(2.10)\quad \operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z_0)}-(p+1)\right\}$$
$$<\frac{(p-\alpha)-2(p(n+p-1)+\alpha)(c+1-\alpha)}{2(n+p)(c+1-\alpha)}$$

for $0\leq\alpha<p$, $n>-p$, and $c>p-1$. Then

$$(2.11)\quad F(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)\,dt$$

belongs to $M_{n+p-1}(\alpha)$.

Proof. From the definition of $F(z)$, we have

$$(2.12)\quad z(D^{n+p-1}F(z))'=(c-p+1)D^{n+p-1}f(z)-(c+1)D^{n+p-1}F(z).$$

Using (2.12) and the identity (2.5), the condition (2.10) may be written as

$$(2.13)\quad \operatorname{Re}\left\{\frac{(n+p+1)\dfrac{D^{n+p+1}F(z)}{D^{n+p}F(z)}-(n+2p-c)}{(n+p)-(n+2p-c-1)\dfrac{D^{n+p-1}F(z)}{D^{n+p}F(z)}}-(p+1)\right\}$$
$$<\frac{(p-\alpha)-2(p(n+p-1)+\alpha)(c+1-\alpha)}{2(n+p)(c+1-\alpha)}.$$

We have to prove that (2.13) implies the inequality

$$(2.14)\quad \operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-(p+1)\right\}<-\frac{p(n+p-1)+\alpha}{n+p}.$$

Define $w(z)$ in U by

$$(2.15)\quad \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}-(p+1)$$
$$=-\left\{\frac{p(n+p-1)+\alpha}{n+p}+\frac{p-\alpha}{n+p}\frac{1-w(z)}{1-w(z)}\right\}.$$

Clearly $w(z)$ is regular and $w(0)=0$. The equation (2.15) may be written as

$$(2.16)\quad \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}=\frac{(n+p)+(n+3p-2\alpha)w(z)}{(n+p)(1+w(z))}.$$

Differentiating (2.16) logarithmically and simplifying we obtain

$$(2.17)\quad \frac{(n+p+1)\dfrac{D^{n+p+1}F(z)}{D^{n+p}F(z)}-(n+2p-c)}{(n+p)-(n+2p-c-1)\dfrac{D^{n+p-1}F(z)}{D^{n+p}F(z)}}-(p+1)$$
$$=-\left\{\frac{p(n+p-1)+\alpha}{n+p}+\frac{(p-\alpha)}{(n+p)}\frac{1-w(z)}{1+w(z)}\right\}$$
$$+\frac{2(p-a)z\,w'(z)}{(n+p)(1+w(z))[(c+1-p)+(p-2\alpha+c+1)w(z)]}.$$

The remaining part of the proof is similar to that of Theorem 1.

Putting $p=c=1$ and $n=\alpha=0$ in Theorem 2, we obtain the following result obtained by Goel and Sohi [3] and Ganigi and Uralegaddi[2].

Corollary 1. *If* $f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k$ *and satisfies the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<\frac{1}{4},$$

then

$$F(z)=\frac{1}{z^2}\int_0^z tf(t)\,dt$$

belongs to Σ^* (*the class of meromorphically starlike functions*).

Remark 1. Corollary 1 extends a result of Bajpai [1].

Theorem 3. *If* $f(z)\in M_{n+p-1}(\alpha)$, *then*

$$F(z)=\frac{n+p}{z^{n+2p}}\int_0^z t^{n+2p-1}f(t)\,dt$$

belongs to $M_{n+p}(\alpha)$.

Proof. For

$$F(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)\,dt,$$

we have

$$(c-p+1)D^{n+p-1}f(z)=(n+p)D^{n+p}F(z)-(n+2p-c-1)D^{n+p-1}F(z)$$

and

$$(c-p+1)D^{n+p}f(z)=(n+p+1)D^{n+p+1}F(z)-(n+2p-c)D^{n+p}F(z).$$

Taking $c=n+2p-1$ in the above relations we obtain

$$\frac{(n+p+1)D^{n+p+1}F(z)-D^{n+p}F(z)}{(n+p)D^{n+p}F(z)}=\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)},$$

which reduces to

$$\frac{(n+p+1)D^{n+p+1}F(z)}{(n+p)D^{n+p}F(z)} - \frac{1}{n+p} = \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}.$$

Thus

$$\mathrm{Re}\left\{\frac{(n+p+1)D^{n+p+1}F(z)}{(n+p)D^{n+p}F(z)} - \frac{1}{n+p} - (p+1)\right\}$$
$$= Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (p+1)\right\} < -\frac{p(n+p-1)+\alpha}{n+p},$$

from which it follows that

$$\mathrm{Re}\left\{\frac{D^{n+p+1}F(z)}{D^{n+p}F(z)} - (p+1)\right\} < -\frac{p(n+p)+\alpha}{n+p+1}.$$

This completes the proof of Theorem 3.

Remark 2. Taking $p = 1$ and $\alpha = 0$ in the above theorems, we get the results obtained by Ganigi and Uralegaddi [2].

References

[1] S. K. Bajpai: A note on a class of meromorphic univalent functions. Rev. Roum. Math. Pures Appl., **22**, 295–297 (1977).

[2] M. D. Ganigi and B. A. Uralegaddi: New criteria for meromorphic univalent functions. Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), **33** (81), no. 1, 9–13 (1989).

[3] R. M. Goel and N. S. Sohi: On a class of meromorphic functions. Glas. Mat., **17**, 19–28 (1981).

[4] I. S. Jack: Functions starlike and convex of order α. J. London Math. Soc., (2) **3**, 469–474 (1971).

[5] V. Kumar and S. L. Shukla: Certain integrals for classes of p-valent meromorphic functions. Bull. Austral. Math. Soc., **25**, 85–97 (1982).

[6] St. Ruscheweyh: New criteria for univalent functions. Proc. Amer. Math. Soc., **49**, 109–115 (1975).

[7] B. A. Uralegaddi and M. D. Ganigi: Subclasses of p-valent meromorphic starlike functions. J. Math. Res. Exposition, **9**, no. 3, 397–401 (1989).