50. *The Generalized Divisor Problem in Arithmetic Progressions*

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Let $d_z(n)$ be a multiplicative function defined by

$$\zeta^z(s)=\sum_{n=1}^{\infty}\frac{d_z(n)}{n^s}\quad(\sigma>1)$$

where $s=\sigma+it$, z is a complex number, and $\zeta(s)$ is the Riemann zeta function. Here $\zeta^z(s)=\exp(z\log\zeta(s))$ and let $\log\zeta(s)$ take real values for real $s>1$.

The following asymptotic formula was considered by G. J. Rieger [5], which is a generalization of Theorem 1 of A. Selberg [6]:

$$D_z(x,q,l)\equiv\sum_{\substack{n\leq x\\ n\equiv l(\mathrm{mod}\ q)}}d_z(n)=\left(\frac{\varphi(q)}{q}\right)^z\frac{x}{\Gamma(z)\varphi(q)}(\log x)^{z-1}+O\left(\left(\frac{\varphi(q)}{q}\right)^z\frac{x}{\varphi(q)}(\log x)^{\Re z-2}\log\log 4q\right)\tag{1}$$

uniformly for $|z|\leq A$, $q\leq(\log x)^\tau$, $(q,l)=1$, where A and τ are any fixed positive numbers.

Next, let $\pi_k(x)$ be the number of integers $\leq x$ which are products of k distinct primes. For $k=1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x)=\frac{xQ(\log\log x)}{\log x}+O\left(\frac{x(\log\log x)^k}{k!(\log x)^2}\right)\tag{2}$$

uniformly for $1\leq k\leq A\log\log x$, where $Q(x)$ is polynomial of degree $k-1$.

Now we define $\pi_k(x,q,l)$ as a generalization of $\pi_k(x)$ by

$$\pi_k(x,q,l)\equiv\sum_{\substack{n\leq x\\ n\equiv l(\mathrm{mod}\ q)\\ n\equiv p_1\cdots p_k(p_i\neq p_j)}}1.$$

In this paper we shall consider the connections between the asymptotic formulas of $D_z(x,q,l)$, $\pi_k(x,q,l)$ and the location of zeros of the Dirichlet L-function. In particular we shall establish some necessary and sufficient conditions for the truth of the Riemann hypothesis, so that this paper gives a generalization of [1] to arithmetic progressions.

The main term of (1) and (2) is, however, inconvenient for our aim so that we introduce the following integrals as the main terms of $D_z(x,q,l)$ and $\pi_k(x,q,l)$ respectively:

$$\Phi_z(x, q) = \frac{1}{2\pi i}\int_L (L(s, \chi_0))^z \frac{x^s}{s} ds,$$

$$F_{k,\delta}(x, q) = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{L_\delta} (L(s, \chi_0))^z \times \left\{\prod_p \left(1 + \frac{z\chi_0(p)}{p^s}\right)\left(1 - \frac{\chi_0(p)}{p^s}\right)^z\right\} \frac{1}{z^{k+1}} \frac{x^s}{s} ds\, dz$$

where L is, for any $r (0 < r < 1/2)$, the path which begins at $1/2$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2$ along the real axis, and L_δ is, for every δ and any r $(\delta > 0, r > 0, \delta + r < 1/2)$, the path which begins at $1/2 + \delta$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2 + \delta$ along the real axis. Here we denote by χ_0 the principal character *mod q*.

The error terms are defined by

$$\Delta_z(x, q, l) = D_z(x, q, l) - \frac{1}{\varphi(q)} \Phi_z(x, q),$$

$$R_{k,\delta}(x, q, l) = \pi_k(x, q, l) - \frac{1}{\varphi(q)} F_{k,\delta}(x, q).$$

Let

$$\Theta(\chi) = \sup\{\sigma : L(\sigma + it, \chi) = 0\}, \quad \Theta_q = \max_{\chi (mod q)} \Theta(\chi).$$

Theorem 1. *There exists some constant c such that*

$$\Delta_z(x, q, l) \ll xe^{-c\sqrt{\log x}}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$ *where* A *and* τ *are any fixed positive numbers.*

Further we have

$$\Delta_z(x, q, l) \ll x^{\Theta_q+\varepsilon}$$

uniformly for $|z| \leq A$, $q \leq x$, $(q, l) = 1$.

Conversely if $\Delta_z(x, q, l) \ll x^{\Xi+\varepsilon}$ *for any* l $((q, l) = 1)$ *and for some* $z \in C - Q^+$, *where* Q^+ *denotes the set of all non negative rational numbers, then any* $L(s, \chi)$ *(mod q) has no zeros for* $\sigma > \Xi$.

The main term $\Phi_z(x, q)$ *has an asymptotic expansion*

$$\Phi_z(x, q) = x(\log x)^{z-1}\sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z - m)} + O(x(\log x)^{\Re z-N-1})$$

uniformly for $|z| \leq A$. *Here* N *is any fixed positive integer and* $B_m(z, q)$ $(0 \leq m \leq N - 1)$ *are regular functions of z, especially* $B_0(z,q) = (\varphi(q)/q)^z$.

Theorem 2. *There is some constant c such that*

$$R_{k,\delta}(x, q, l) \ll xe^{-c\sqrt{\log x}}$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q, l) = 1$.

Further we have

$$R_{k,\delta}(x, q, l) \ll x^{\Theta_q+\varepsilon}$$

uniformly for $k \geq 1$, $q \leq x$, $(q, l) = 1$.

Conversely if $R_{k,\delta}(x, q, l) \ll x^{\Xi+\varepsilon}$ *for any* $l((q, l) = 1)$ *and for some* $k \geq 1$, *then any* $L(s, \chi)$ *(mod q) has no zeros for* $\sigma > \Xi$.

The main term $F_{k,\delta}(x, q)$ *has an asymptotic expansion*

$$F_{k,\delta}(x, q) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O\left(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every k and q. Here N is any fixed positive integer and $Q_{m,q}(x)$ are polynomials of degree not exceeding $k-1$, especially the coefficient of x^{k-1} of $Q_{0,q}(x)$ is 1.

Remark. 1. If we define $r_{k,q,l}$ by

$$r_{k,q,l} = \inf_{\delta} \inf\{r : R_{k,\delta}(x, q, l) \ll x^r\}$$

Theorem 2 shows that $r_{k,q,l} = \Theta_q$. The statement $\Theta_q = 1/2$ for every q is equivalent to the truth of the Riemann hypothesis for Dirichlet L-function.

2. For $k=1$, we can express the main term in terms of the logarithmic integral. Namely,

$$F_{1,\delta}(x, q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}),$$

so that

$$\pi_1(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{du}{\log u} + O(xe^{-c\sqrt{\log x}}).$$

3. Similar results hold for $\omega_k(x, q, l)$ and $\Omega_k(x, q, l)$. Here

$$\omega_k(x, q, l) \equiv \sum_{\substack{n \leqq x \\ n \equiv l (\mathrm{mod}\, q) \\ \omega(n) = k}} 1, \quad \Omega_k(x, q, l) \equiv \sum_{\substack{n \leqq x \\ n \equiv l (\mathrm{mod}\, q) \\ \Omega(n) = k}} 1$$

where $\omega(n)$ means the number of distinct prime factors of n, and $\Omega(n)$ means the number of total prime factors allowing multiplicity.

The proof of Theorems 1 and 2 goes on similar lines as those of [1], using well-known zero free region for $L(s, \chi)$. The detail will apear in [2].

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